YOBE STATE UNIVERSITY, DAMATURU
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
MATHS1201 (ELEMENTARY ALGEBRA AND SET)
LECTURE NOTE

COURSE OUTLINE

(1) **Set theory:** Definition of set, types, relationship between sets, Venn diagram, Relation & function

(2) **Real numbers:** Rational and Irrational numbers, Real number lines and Inequalities.

(3) **Polynomials:** Factorization, Reminder and factor theorem, Quadratic equations, Rational function & Partial fraction decomposition, Graph of polynomials.

(4) **Linear equations:** Linear equations & consistency, matrices & determinants of 2^{nd} & 3^{rd} order.

(5) **Sequence and series:** types of sequence and summation of simple series.

(6) Mathematical induction and binomial theorem.

SET THEORY

In our real life and everyday activities, we admire an arrangement or collection of things according to their common property. For instance: teachers always arrange their students name alphabetically. And people normally arrange their shoes and dresses accordingly, so each of the two arrangements above is termed as a set.

Definition: A set can be define as a well define collection of an object according to a common property, these objects can be anything (i.e numbers, words, items etc) and its called the elements or members of the set.

A set is always denoted by a capital letters & its elements or the members are denoted by small letters

Example of sets: (i) A $= \{1,2,3, \cdots \infty\}$ (ii) P $= \{ names, articles, lines, letters \}$ (iii) Z $= \{a, b\}$.

DESCRIPTION AND CADINALITY OF SETS:

(i) By listing method or tabular form (Roster). Example: W = $\{ sat, sun, mon, tue, wed, thr, fri \}$

(ii) By stating a rule, set builder or property form. Example: W = $\{ days\ of\ the\ week \}$, V = $\{ vowels \}$ and C = $\{ x: 1 < x < 10\ \ \& x\ is\ an\ integer \}$. While the **Cardinality of a set** means the number of elements in a set, and it is denoted as n(set). Example: Let A $= \{1,2,3\}$, then n(A) $= 3$.

EXERCISES (1)

(1) Write the below sets in a tabular form and state their cardinalities also ? (a) A = $\{ a\colon\ a^2 = 4 \}$ (b) B = $\{ b: b - 2 = 15 \}$ (c) C = $\{ c: c\ is\ positive\ no's\ < 2\ 0 \}$ (d) D = $\{ all\ HODs\ of\ YSU \}$.

TYPES OF SET:

(1) **Finite set:** this is a set which has a definite (countable) number of elements. Examples: (a) Q = $\{ 1, 2, 3, 4, 5 \}$ $\Longrightarrow$ n(Q) = 5. (b) X = $\{ x: 1 < x < 10 \} = \{ 2, 3, 4, \cdots, 9 \}$ $\Longrightarrow$ n(X) = 9.
(c) A states in the north east zone of Nigeria, i.e N = $\{ Borno, Yobe, Bauchi, Gombe, Yola, Taraba \}$.

(2) **Infinite set:** this is a set whose elements are uncountable (i.e the elements have no limit) Example (a) A= $\{no\ of\ stars\ in\ the\ sky\}$ $\Longrightarrow$ n(A) = ∞. (b) Y= $\{y: y\ is\ an\ odd\} = \{1,3,5, \cdots\}$ $\Longrightarrow$ n(Y)= ∞

(3) **Null or Empty set:** this is a set without any elements in it, it is denoted as $\{\ \}$ or $\emptyset$. Example (a) B= $\{triagle\ with\ 5\ vertices\}$ $\Longrightarrow$ n(B) = $\emptyset$. (b) P= $\{all\ students\ with\ 4\ legs\}$ $\Longrightarrow$ n(B) = $\emptyset$. (c) G = $\{ g: g^2 = 4, g\ is\ an\ odd\ no' \} = \{\ \}$ $\Longrightarrow$ n(G) = $\emptyset$. The solution is 2 & -2 and none of the $no's$ is odd, therefore G = $\emptyset$. Note: $\{ 0 \}$ this is not an empty set, but rather a singleton set. Therefore a"**Singleton set**" is a set with only one element in it. Example: let H = $\{h: h^2 = 1\} = \{1\}$ $\Longrightarrow$ n(H) = 1

(4) **Equal set:** two sets are said to be equal if they have the same elements. Example (a) A = $\{ 1, 2, 3, 4 \}$ & B = $\{ 3, 1, 4, 2 \}$ $\Longrightarrow$ A = B. (b) P = $\{ 1, 2, 3, 4 \}$ & Q= $\{ 3, 2,1, 4, 3,2 \}$ $\Longrightarrow$ P = Q. (c) S = $\{ first\ five\ letters\ of\ the\ alphabet \}$ & R = $\{ a, b, c, d, e \}$ $\Longrightarrow$ S = R.

(5 **Equivalent set:** Two sets are said to be equivalent if they have equal number of elements. **Example:** (a) A = $\{ 1, 2, 3, 4 \}$ & B = $\{ a, b, c, d \}$ $\Longrightarrow$ A ≡ B. (b) S = $\{ first\ five\ letters\ of\ the\ alphabet \}$ & R = $\{ a, b, c, d, e \}$ $\Longrightarrow$ S ≡ R. **Note that:** all equal sets are equivalent but not all equivalent sets are equal. (that is not vice versa). And the symbol used for equivalence is '≡'.

(6) **Disjoint set:** two sets are said to be disjoint if they have no elements in common. **Example** (a) A = $\{ 1, 2, 3, 4 \}$ & B = $\{ 6, 7, 8, 9 \}$ $\Longrightarrow$ A ≠ B. (b) S = $\{ first\ five\ letters\ of\ alphabet \}$ & R = $\{ a, e, i, o, u \}$ $\Longrightarrow$ S ≠ R.

(7) **Subset:** this can be define as a set within another set. Any set which is a part of a larger is a subset. **Example:** A = $\{ 1, 2, 3, \}$, B = $\{ 1, \}$, C = $\{ 1, 2 \}$ & D = $\{ 1, 2, 3\}$. So each of these sets B, C & D is a subset of the set A. There are basically two types of subsets:

(i) **Proper Subset:** this is a set which is not the set itself, and it has a symbol ⊂ (which means " is a subset of " or "contained in"), and ⊃ (which means " it contains" or " has as one of its subsets "). **Example:** If A = $\{1, 2, 3, \}$ & B = $\{ 2, 3 \}$, then A is a proper subset of B, because B ⊂ A or A ⊃ B.

(ii) **<u>Improper Subset:</u>** this is a set which is the set itself, and it has a symbol $\subseteq$ (which means " is a subset and equal to"), and $\supseteq$ (which means " it contains and equal to").

Example. If $A = \{ 1, 2, 3 \}$ & $B = \{ 1, 2, 3 \}$, or $B = \{ 3, 2, 1 \}$, then B is an improper subset of A, because $A \subseteq B$ or $A \supseteq B$. And $B \subseteq A$ or $B \supseteq A$.

(8) **<u>Universal Set:</u>** this is a set containing all the elements under discussion in a particular problem, and the sets under discussion can be regarded as the subsets. Universal set is denoted by $\cup$ or £.

Example: $\cup$ or £ ={ students of Y.S.U }, $\cup$ ={ states in a country }, £ ={ letters of the alphabets }

(9) **<u>Power Set:</u>** this is a family of all the subsets of a give set. It is denoted by 2^n where n is the number of elements. Note: (1) every set is a subset of itself. (2) Empty set is a subset of every set.

Example. If $A = \{ 1, 2, 3, \}$, then $n(A) = 3$, and then the power set is $2^3 = 8$, they are: $\{ 1, 2, 3, \}$, $\{ \quad \}, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 1, 2, \}, \{ 1, 3 \}, \{ 2, 3 \}$.

EXERCISES (2)

(1) Given the set $A = \{ 1, 2, 3, 4 \}$, $B = \{ 1, 3, 5, 7\}$, $C = \{ 1, 2, 3, 4, 5\ 6, 7, 8, 9, 10 \}$, find how many subsets do they contain and list all the subsets of the set B ?

(2) Given the set $A = \{ 1, 2, 3, 4 \}$, $B = \{ 1, 3, 5, 7\}$, $C = \{ 1, 2, 3, 4, 5\ 6, 7, 8, 9, 10 \}$, State which of the following is true or false ? (a) $A \subset C$ (b) $B \subset C$ (c) $A \not\subset C$ (d) $B \not\subset C$

(3) which of these sets are equal and state the reason why it is an equal set

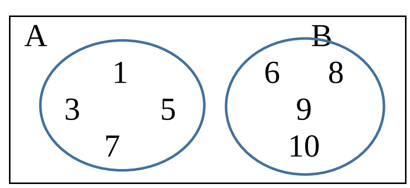
(a) $A = \{ x\text{: } x \textit{ is a letter of the word follow} \}$ (b) $B = \{ y\text{: } \textit{yis a letter of the word flow} \}$

(c) $C = \{ c\text{: } c \textit{ is the letter which appear in the word wolf} \}$ (d) $D = \{ \textit{the letter } f, l, o, w \}$

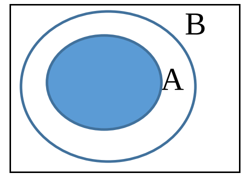
RELATIONSHIP BETWEEN SETS AND VENN DIAGRAM.

A Venn diagram was named after John Venn and is used to illustrate sets operation with its different relationship. The Venn diagram comprises of circles (which represents the subsets) enclosed in a rectangle (which represent the universal set).

Example (1): Let $\cup = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$, $A = \{ 1, 3, 5, 7\}$ & $B = \{ 6, 8, 9, 10 \}$. Here both A & B represent subsets and they were exhibited in a circle inside a rectangle as stated above.

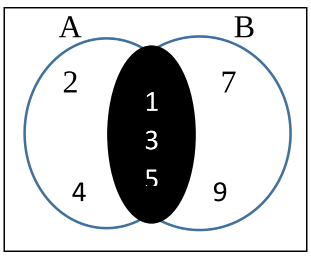


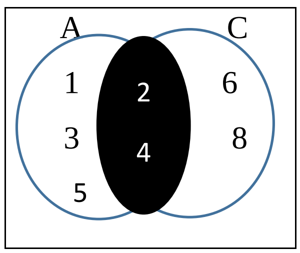
Example (2): If $A \subset B$, and $A \subseteq B$, that is A is a subset of B, and B is a superset of A, we can also use Venn diagram to represent the given information.

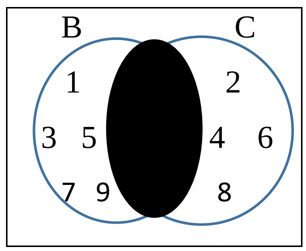


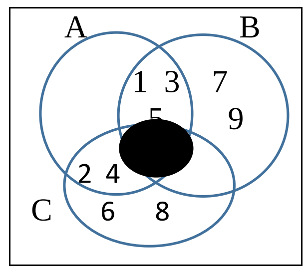
So below are full details on relationship between sets (Operations with sets), & the basic operations are (1) **<u>Intersection of Sets:</u>** this is defined as a set that contained all the common elements between two or more sets, and it is denoted by the symbol ∩.

Example (1): If $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 7, 9\}$, and $C = \{2, 4, 6, 8\}$, then (a) $A \cap B = \{1, 3, 5\}$ (b) $A \cap C = \{2, 4,\}$ (c) $B \cap C = \{\ \}$. (d) $A \cap B \cap C = \{2, 4,\}$ These information can also be represented using Venn diagram

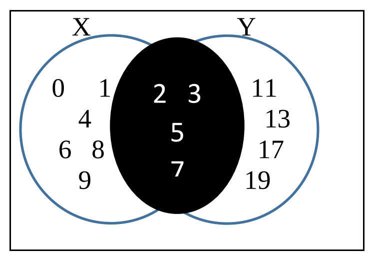




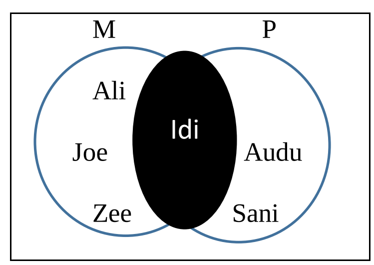




<u>Example(2):</u> Given that $X = \{whole\ no's < 10\} = \{0, 1, \cdots, 9\}$, and $Y = \{prime\ no's < 20\} = \{2, 3, \cdots, 19\}$ then $X \cap Y = \{2, 3, 5, 7\}$. These information can also be represented using Venn diagram

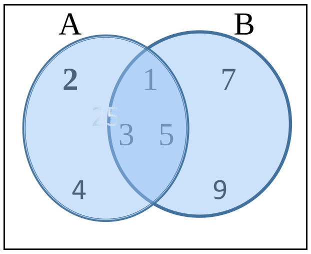


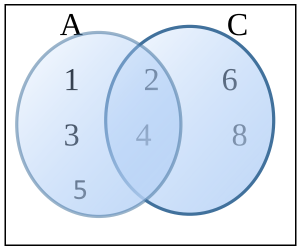
Example (3): In a school, the principal wants to know the number of students offering both Maths and Physics in S.S.3. If M = { Ali, Idi, Joe, Zee } and P = { Audu, Idi, Sani }, then $M \cap P = \{Idi\}$. Here the shaded part is the intersection (that is only one person offering the both subjects.

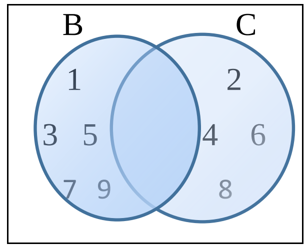


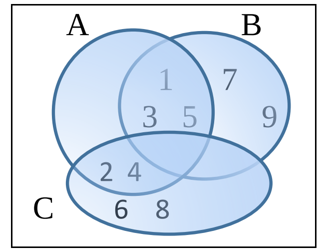
(2) **Union of set:** this is a set that contained all the elements of two or more sets without repetition, and it is denoted by the symbol ∪.

<u>Example (1):</u> If $A = \{1, 2, 3, 4.5\}$, $B = \{1, 3, 5, 7, 9\}$, and $C = \{2, 4, 6, 8\}$, then (a) $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$ (b) $A \cup C = \{1, 2, 3, 4, 5, 6, 8\}$ (c) $B \cup C = \{1, 2, 3, 5, 6\}$. (d) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\}$. It can also be represented by Venn diagram

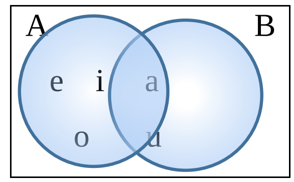




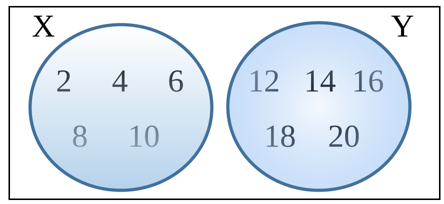




Example (2) If $\mathrm{A} = \{ a, e, i, o, u \}$, $\mathrm{B} = \{ a, t, y, s, u \}$, then $\mathrm{A} \cup \mathrm{B} = \{a, e, i, o, t, y, s, u\}$, this information can also be represented by a Venn diagram



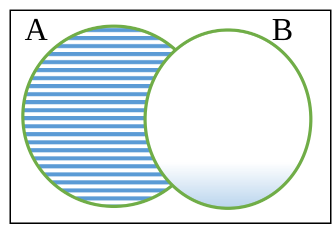
Example (3) Let $\mathrm{X} = \{ 2, 4, 6, 8, 10 \}$, $\mathrm{Y} = \{ 12, 14, 16, 18, 20 \}$, then $\mathrm{X} \cup \mathrm{Y} = \{ 2, 4, 6, 8, 10, 12, 14, 16, 18, 20 \}$, and this information can also be represented using the Venn diagram.



(3) **Difference of two Sets:** the difference of two sets (A & B) are the elements which belong to set A but don't belong to set B & vice versa. It is denoted by the symbol $A - B$

Example (1). If $\mathrm{A} = \{ a, e, i, o, u\}$, $\mathrm{B} = \{a, t, y, s, u\}$, then $\mathrm{A} - \mathrm{B} = \{e, i, o\}$, and this information can also be represented using the Venn diagram.

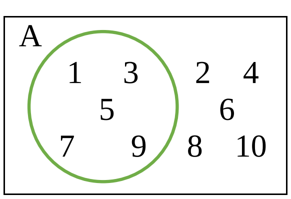
Therefore the shaded part represents $\mathrm{A} - \mathrm{B}$. Note that $\mathrm{A} - B \neq B - A$.

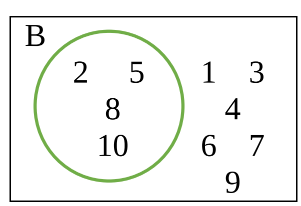


(4) **<u>Complement of a Sets:</u>** the complement of a set can be clearly understood if we relate it to the universal set. For instance. If $\cup = \{a, e, i, o, u \}$, $\mathrm{A} = \{ a, e, u \}$, then $A \subset \cup$. Now the **"Complement of a Sets"** can be defined as a set of elements that do not belong to the set A but contained in the universal set $\cup$. The complement of a set is just the difference of universal set with the set A. It is denoted by A' or A^c.

Example (1). If $\cup = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$, $\mathrm{A} = \{ 1, 3, 5, 7, 9 \}$, $\mathrm{B} = \{ 2, 5, 8, 10 \}$, then we have $A' = \{ 2, 4, 6, 8, 10 \}$ $B^c = \{ 1, 3, 4, 6, 7, 9 \}$, **Note**: (i) A^c is define as the set $\{ x: x \in \cup \ \& \ x \notin A \}$ (ii) $A \cup A^c = \cup$ (iii) $A \cap A^c = \emptyset$ (iv) $\cup^c = \emptyset$ and vice versa.

$\cup$ =





Example (2): Suppose $\bigcup = \{ \, set\ of\ natural\ no's \, \}$ and $P = \{ \, odd\ no's \, \}$ then $P^c = \{ \, even\ no's \, \}$.

EXERCISES (3)

(1) Let $\bigcup = \{ 1, 2, 3, \cdot\cdot\cdot, 10 \}$, $P = \{ 2, 3, 5, 7, 11, 13, 17, 19, 23 \}$, $Q = \{2, 4, 6, 8, 10, \cdot\cdot\cdot, 30\}$ and $R = \{ 1, 3, 5, 7, \cdot\cdot\cdot, 29 \}$, Find the following by listening the elements.

(a) $(P \cap Q)$ (b) $(P \cup Q$ (c) $(P \cap Q \cap R)$ (d) $(P \cup Q \cup R)$ (e) $(P - Q)$ (f) $(Q - R)$ (g) $(P - R)^c$ (h) $P \cup (Q \cap R)^c$ (i) $(P \cap Q)^c - R$. (J) Represent the information obtained (that is your answers) using Venn diagram.

(2) Let $\bigcup = \{ 1, 2, 3, \cdot\cdot\cdot, 10 \}$, $P = \{ 2, 3, 5, 7, 11, 13, 17, 19, 23 \}$, $Q = \{ 2, 4, 6, 8, 10, \cdot\cdot\cdot, 30\}$ and $R = \{ 1, 3, 5, 7, \cdot\cdot\cdot, 29 \}$, Find the following by listening the elements.

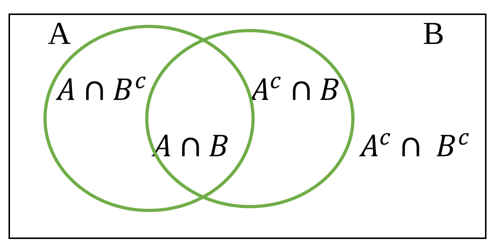
(a) P^c (b) B^c (c) C^c (d) $(P \cup Q)^c$ (e) $(P \cap R)^c$ (f) $(P \cup Q \cup R)^c$ (g) $(P \cap Q \cap R)^c$ (h) $\{ P - Q \}^c$ (i) $\{ Q - R \}^c$ (j) $\{ P - R \}^c$ (k) $P \cup (Q \cap R)^c$ (l) $(P \cap Q)^c - R$ (M) Represent the information obtained (that is your answers) using Venn diagram.

(3) Given that $A = \{ x : x \in N,\ and\ x < 25 \}$, If $B = \{ even\ no's \}$, $C = \{ perfect\ squares \}$ and $D = \{ numbers\ divisible\ by\ 4 \}$, then solve for the following problems using the above information: (i) $n(B)$ (ii) $n(C)$ (iii) $n(D)$ (iv) $B \cap C$ (v) $B \cup D$ (vi) $B^c \cap C^c$ (vii) $(B \cap D)^c \cap C$ (viii) D^c (ix) Represent the information obtained (that is your answers) using Venn diagram.

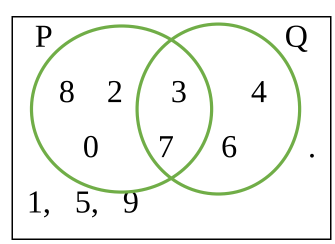
APPLICATION OF A VENN DIAGRAM (TWO AND THREE SET PROBLEMS)

Word problems involving more than one set at a time can easily be solved by the use of Venn diagram, In both two and three set problems, it is always good to get a clear picture of the different sections of the venn diagram, and what is represents also. For instance,

$\bigcup =$



Example (1): the Venn diagram below represents a universal set U of integers and its subsets P & Q, then list the elements of the following sets: (i) $P \cup Q$ (ii) $P \cap Q$ (iii) $(P \cup Q)^c$ (iv) $(P \cap Q)^c$. ?



Sln: $U = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \}$ $P = \{ 0, 2, 3, 7, 8, \}$ $Q = \{ 3, 4, 6, 7, \}$ (i) $P \cup Q = \{ 0, 2, 3, 4, 6, 7, 8 \}$ (ii) $P \cap Q = \{ 3, 7 \}$ (iii) $(P \cup Q)^c = \{ 1, 5, 9 \}$ (iv) $(P \cap Q)^c = \{ 0, 1, 2, 4, 5, 6, 8, 9 \}$

Example (2): On a matriculation day of new students in YSU, 800 students turned up at the opening ceremony, 600 students tuned up at the departmental Orientation and there is a total of 1234 student altogether in the University, how many students attended both activities, and represent your answer using Venn diagram ?

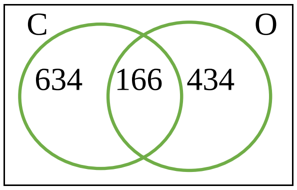
Solution: Here Our $n(U) = 1234$, $n(C) = 800$, $n(O) = 600$, then $n(C \cap O) = x$?

To find this easily we use the rule $n(C \cup O) = n(C) + n(O) - n(C \cap O)$,

so we have $1234 = 800 + 600 - x$

$1234 = 1400 - x$

$-166 = -x$. So $x = 166$ $\therefore$ 166 students attended both activates



Example (3): A recent survey of 40 students in a class revealed that 6 study Maths (M) & Physics (P), 9 study Physics (P) & Chemistry (C), 7 study Maths (M) & Chemistry (C), 15 study Maths (M), 22 study Chemistry (C), 18 study Physics (P) and 4 study all the three subjects. Then find how many students study (a) C only (b) M only (c) P & C only (d) M & P only (e) don't study any of the three

Solution: Here $n(M) = 15$, $n(C) = 22$, $n(P) = 18$, then we have the following using the diagram

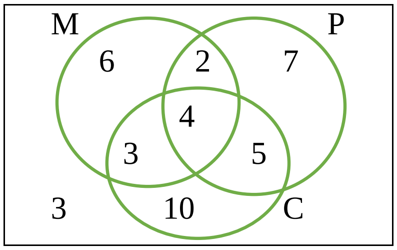
(a) Chemistry only = $n(C \cap M^c \cap P^c) = 22 - (3 + 4 + 5) = 10$

(b) Maths only = $n(M \cap P^c \cap C^c) = 15 - (2 + 4 + 3) = 6$

(c) P & C only = $n(P \cap C \cap M^c) = 9 - 4 = 5$

(d) M & P only = $n(M \cap P \cap C^c) = 6 - 4 = 2$

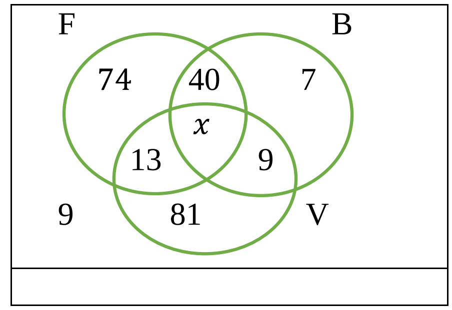
(e) $n(M^c \cap C^c \cap P^c) = 40 - (6 + 2 + 7 + 4 + 3 + 5 + 10) = 3$



Example (4): A sport club has facilities for football (F), basketball (B) and Volleyball (V). An inquiry into the use of these facilities by the 274 members revealed the following results: $n(F) = 130$. $n(B) = 59$, $n(V) = 106$, $n(F \cap B) = 40$, $n(B \cap V) = 9$, $n(V \cap F) = 13$. If 38 members do not use any of the facilities at all ? Then find (a) how many members use all the facilities (b) Determine (i) $n(F \cap V \cap B^c)$ (ii) $n(F \cap B \cap V^c)$ (iii) $n(F^c \cap B \cap V)$.

Solution: Here our $n(U) = 274 - 38 = 236$, and $n(F \cap B \cap V) = x$?

(a) To find $n(F \cap B \cap V)$ we use the rule $n(F \cup B \cup V) = n(F) + n(B) + n(V) - n(F \cap B) - n(F \cap V) - n(B \cap V) + n(F \cap B \cap V)$. $\therefore$ we have $236 = 130 + 59 + 106 - 40 - 13 - 9 + x \implies 236 = 233 + x \implies 236 - 233 = x \quad \therefore x = 3$



(i) From the diagram $n(F \cap V \cap B^c) = 13 - x = 10$

(ii) From the diagram $n(F \cap B \cap V^c) = 40 - x = 37$

(iii) From the diagram $n(F^c \cap B \cap V) = 9 - x = 6$.

EXERCISES (4)

(1) In an integrated school interhouse Competition, 80% of the students turned up at the Mathematics competition, 60% of the students tuned up at the Arabic competition, find the percentage number of the students who attended both activities, and represent your answer using Venn diagram ?

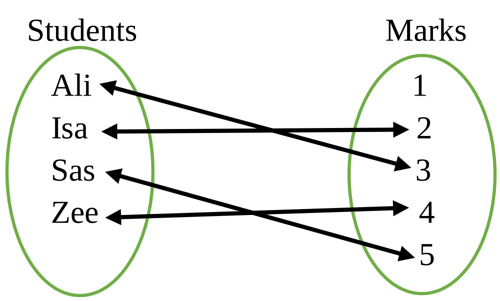
(2) A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked forein rice while 450 liked Local rice, what is the least number of customers that like both product, illustrate the information and your result in a Venn diagram ?

(3) A number of candidates interviewed recently in Yobe State revealed that 1,420 applied for admission into YSU, 1,220 applied for Unimaid, 1,180 applied for B.U.K, 660 had applied for YSU & Unimaid, 480 applied for Unimaid & B.U.K. and 580 applied for YSU & B.U.K while 320 had applied for all the Universities. Then (a) illustrate the information using Venn diagram ? (b) number of candidates interviewed ?

(4) At sport club with 95 members, it was found that 90 played Lawn Tennis, 76 played Table Tennis, 67 played Badminton, 35 played both Lawn & Table Tennis, 28 played both Table Tennis & Badminton and 30 played both Lawn Tennis & Badminton, it was also shown than 35 of them did not play any of the games. Then find (i) how many played all the three games (ii) how many played Lawn Tennis only (iii) how many played Lawn Tennis only

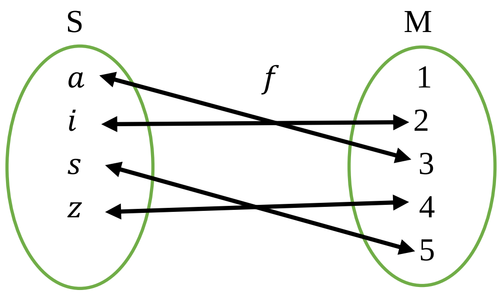
RELATION AND FUNCTION

A relation is a way of establishing a bond between the elements of two or more sets. For instance, Suppose A & B are brothers, we can use double headed arrow ($\leftrightarrow$) to show the linkage between them. Example (1)" four students Ali, Isa, Sas and Zee were given a spelling test over 5, their marks was recorded as shown in an arrow diagram below.



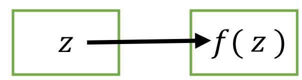
By choosing any name from the above set of students, we can easily find the marks relates to it. So any relationship which takes one element of a first set and assign to a unique element of the second set is called a function. So now, a function simply means a relationship between two or more sets, it is also called as correspondence.

Example (2)



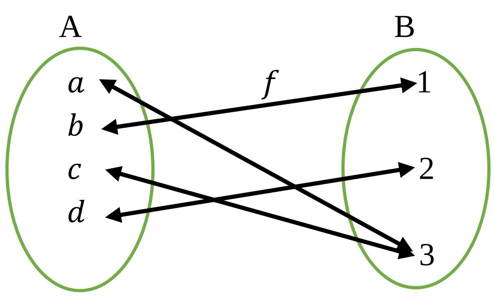
And if the relation is denoted f, then f is said to be a function or mapping, written as $f : S \longrightarrow M$ or $S \xrightarrow{f} M$. the first set S is called the domain and the second set M is called the codomain of the function f. If $z \in S$, the element in M which assigned to z is called the $f - image\ of\ z$, written as $f(z)$.

Then the element a it self is called the $-image\ of\ f(a)$. That is



Note that the set of all $f - images$ in the codomain is called the range of the function f. For instance the domain of the above function is $S = \{a, i, s, z\}$, while the codomain is $M = \{1, 2, 3, 4. 5\}$ and the Range is $R = \{2, 3, 4, 5\}$

Example (3) Let $A = \{a, b, c, d\}$ & $B = \{1, 2, 3,\}$, if $f : A \longrightarrow B$ defined by the arrow diagram below



Then find (1) domain (2) co-domain and the range of the function ?

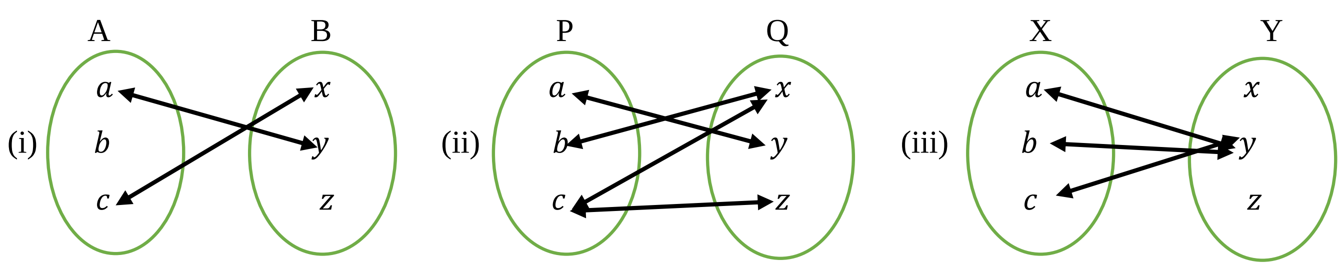
Solution: (i) domain of $f = \{a, b, c, d\}$,

(ii) co-domain of $f = \{1, 2, 3\}$

(iii) Range of $f = \{1, 2, 3\}$.

So in the above example (3) the co-domain and the Range are equal. Note that a function can be a defined and undefined function,

Example (4) Consider the following diagrams & determine whether the function is defined or undefined



(i) is not a defined function, because the element $b \in A$ doesn't assigned to any element in the codomain

(ii) is not a defined function, because the element $c \in P$ is assigned to two elements in the co-domain

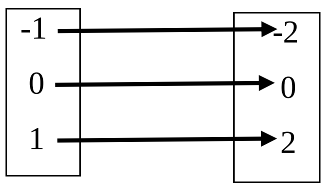
(iii) is a defined function, because every elements in the domain is assigned to an element in the co-domain, and indeed is called a constant function.

TYPES OF FUNCTION

(1) **One – $\boldsymbol{to}$ − One function:** this is a function that every of its elements in the domain is mapped to a distinct elements in the co-domain. It is also called an injective function.

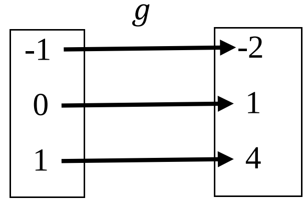
Example: draw an arrow diagram for the function (a) $f : x \longrightarrow 2x$ (b) $g : x \longrightarrow 3x + 1$ with domain (-1, 0, 1) and state the range of the functions ?

Solution: f



Range $= \{ 1, 2, 3 \}$

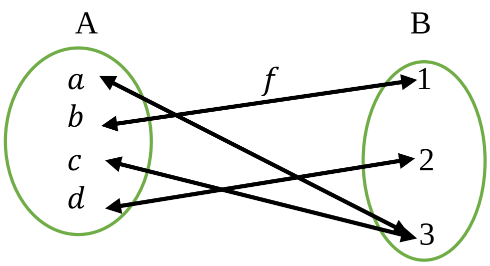
g



Range $= \{-2, 1, 4 \}$

(2) **Onto function:** a function is said to be an onto if every element of the co-domain is also in the Range. It is also called a surjective function.

Example: Let $A = \{ a, b, c, d \}$ and $B = \{ 1, 2, 3, \}$, if $f : A \longrightarrow B$ defined by the arrow diagram below, state whether the function is onto function or not.



Solution: (i) domain of $f = \{ a, b, c, d \}$
(ii) co-domain of $f = \{ 1, 2, 3 \}$
(iii) Range of $f = \{ 1, 2, 3 \}$.

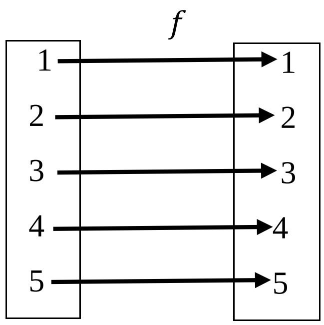
Note that in the above example, the Range is the entire co-domain. Therefore the function is an onto function, because it satisfies the condition aforesaid.

(2) **Identity function:** a function is called an identity function if each element of the first set has an image on itself. That is $f(a) = a \; \forall \; a \in A$. It is denoted by I.

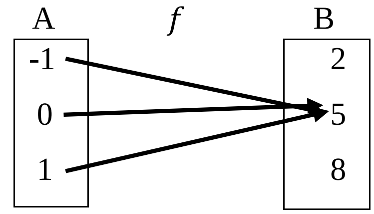
Example (1) Consider $A = \{ 1, 2, 3, 4, 5 \}$ and $f : A \longrightarrow A$ such that $\{ (1,1)(2,2)(3,3)(4,4)(5,5) \}$

Solution: the function is an identity function as every element is mapped onto itself. So the function is both $1 - 1$ and $onto$. And any function that is both $1 - 1$ and $onto$ is also called as bijective function.

Diagrammatically,

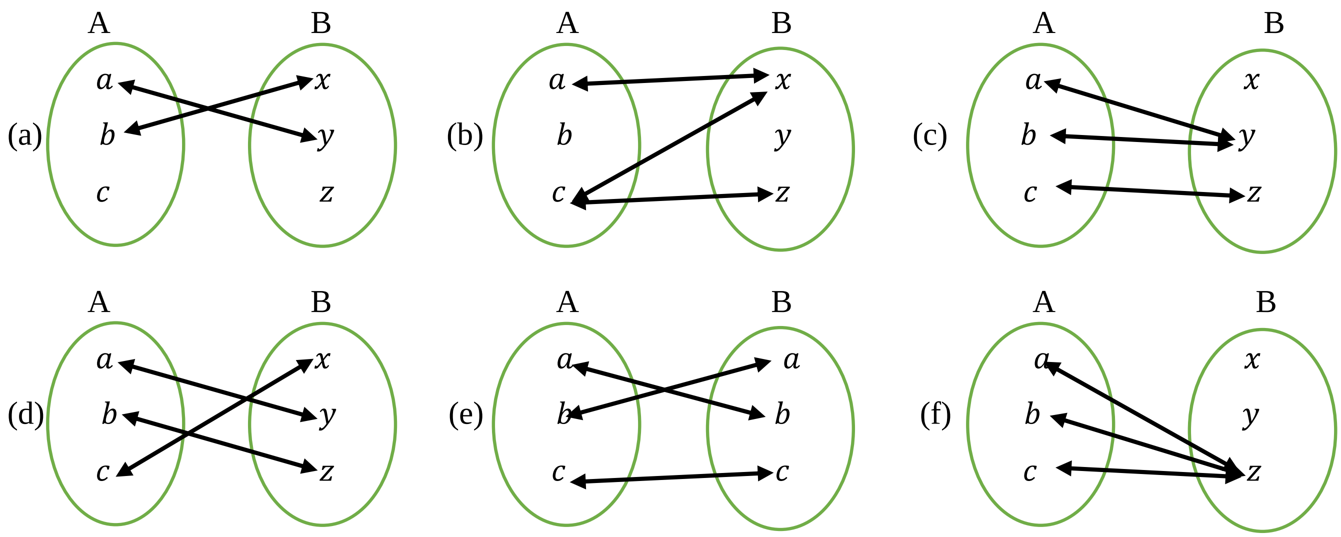


(4) **<u>Constant function:</u>** a function is said to be a constant function if every element in the domain mapped to a single element jn the codomain. For instance



EXERCISES (5)

Study the arrow diagrams below and answer the questions that tailored to the diagrams below it.



(1) State whether each diagrams above defines a function or not from $A = \{ a, b, c \}$ & $B = \{x, y, z\}$

(2) Which of the mappings above is $one - to - one$ or $onto$ from $A = \{ a, b, c \}$ & $B = \{x, y, z\}$

(3) Which of the mappings above is both $one - to - one$ & $onto$ from $A = \{ a, b, c \}$ & $B = \{x, y, z\}$ and show clearly the identity and constant mappings if they exist.

REAL NUMBERS

Definition: A number can be defined as any symbol used to describe a quantity. For instance: $1, 2, 3, \cdots$ or e.t.c, these numbers can be categorized into different classes, such as real numbers, natural numbers e.t.c.

The set of real numbers comprises of all the set of different numbers, and is denoted by the symbol $\mathbb{R}$. The real number system can be categorized as shown below:

(1) Natural number (N): This can be defined as a set of numbers from $1, 2, 3, \cdots, \infty$ (counting no's)

(1a) Odd number$s$: This can be defined as a set of numbers in the form $1, 3, 5, 7, \cdots, \infty$.

(1b) Even number$s$: This can be defined as a set of numbers in the form $2, 4, 6, 8, \cdots, \infty$.

(1c) Prime number$s$: This can be defined as a set of numbers in the form $2, 3, 5, 7, 11, \cdots, \infty$.

(2) Whole number$s$: This can be defined as a set of numbers from $0, 1, 2, 3, \cdots, \infty$ ($that\ is\ 0 + N$)

(3) Integers (Z): This can be defined as a set of numbers from positive and negative together with zero. (that is $Z = -\infty, \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots, \infty$).

(3a) Positive Integers (Z^+): This can be defined as a set of numbers in the form $1, 2, 3, \cdots, \infty$ (N).

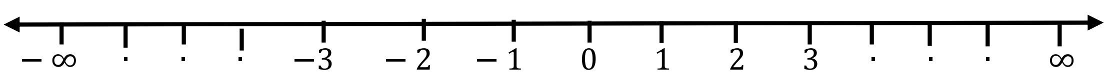
(3b) Negative Integers (Z^-): This can be defined as a set of numbers in the form $-1, -2, -3, \cdots, -\infty$

Note that zero (0) is also an integer but is considered to as neither positive nor negative.

(4) Rational number$s$ (Q): Thise are numbers that can be expressed in the form of fraction: $\frac{1}{2}, -\frac{2}{3}, 8\frac{4}{9}$

(5) Irrational number$s$ (Q'): Thise are numbers that can neither be expressed as fraction nor as terminating decimal, (Q' are square roots of a natural numbers that are not the natural numbers & $Q' = surd$. Examples $\pi, \sqrt{2}, \sqrt{3}, \sqrt{6}$ e.t.c, Pictorially we can write it as $N \subset W \subset Z \subset Q \subset Q' \subset \mathbb{R}$. Generally, numbers are made up of Rational & Irrational number$s$, so the union of these two formed the entire real number system.

REAL NUMBER LINE: This is a straight line showing the ordering property and position of integers:

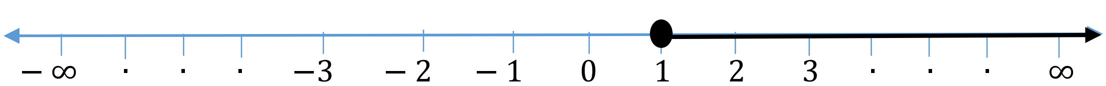


INEQUALITIES

In real life situations, we come across many things that are not equal exactly. For instance $5 \neq 6$. So, we may wish to know which one is smaller or greater than the other than the other. Words like less than or greater than are used in the aspect of inequality, such as $<$, $>$, $\leq$ & $\geq$ and we also used real number line to represent inequalities.

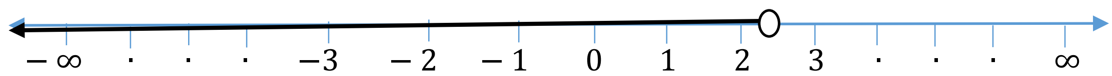
Examples: Solve for the following inequalities & show the results on a number line (i.e graph)

(1) $3x + 7 \geq 8 \implies 3x \geq 8 - 5 \implies 3x \geq 3 \implies x \geq 1$ $\therefore$ The solution contains numbers 1, 2, 3, $\cdots \infty$.



(2) $3x - 5 > 8x - 17 \quad \Rightarrow 3x - 8x > -17 + 5 \quad \Rightarrow -5x > -12 \quad \Rightarrow x < \frac{12}{5}$

∴ The solution contains numbers $(-\infty \cdot \cdot \cdot 2.4)$

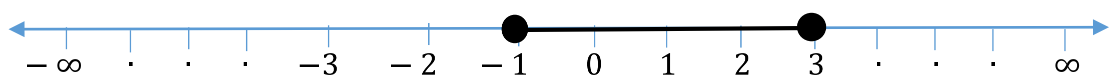


(3) $3 < 2x + 2 \leq 4 \quad \Rightarrow 3 - 2 < 2x + 2 - 2 \leq 4 - 2 \quad \Rightarrow 1 < 2x \leq 2 \quad \Rightarrow ½ < x \leq 1$

∴ The solution contains numbers (½, · · · 1]

(4) $-4 \leq 5 - 3x \leq 8 \quad \Rightarrow -4 - 5 \leq 5 - 3x - 5 \leq 8 - 5 \quad \Rightarrow -9 \leq -3x \leq 3$

$\Rightarrow 3 \leq x \leq -1$ ∴ The solution contains numbers $[-1,\ 0,\ 1,\ 2,\ 3]$



Note: the set of values between two numbers say (a & b) where $a < b$ is called an interval from $a - b$, the numbers a & b are called the endpoint of the interval.

(1) An interval in the form $a < x < b$ is called an open interval, where the endpoints are not included, and it is written as $(a,\ b)$.

(2) An interval in the form $a \leq x \leq b$ is called a closed interval, where the endpoints are included, and it is written as $[a,\ b]$.

(3) And if the other endpoint is not is called a half-open interval, and it is written as $(a,\ b]$ or $[a,\ b)$.

EXERCISE (6)

Simplify the following inequalities and illustrate your solution on a number line or a diagram

(1) $8 - 2x \leq 3$ (2) $4x - 3 \geq 2x + 3$

(3) $-5 < 2x + 3 < 7$ (4) $5 (x + 1) \leq 4x + 1$

(5) $\frac{3}{8}(x - 1) + 2 > \frac{3}{4}(2x + 1) + 1$

(6) $2h - \frac{1}{2} \geq 4\frac{1}{4} - 3h$ (7) $\frac{y}{3} < \frac{y}{5}$ (8) $\frac{k}{3} - \frac{3k}{2} > 7$

Inequalities in Two Dimensions

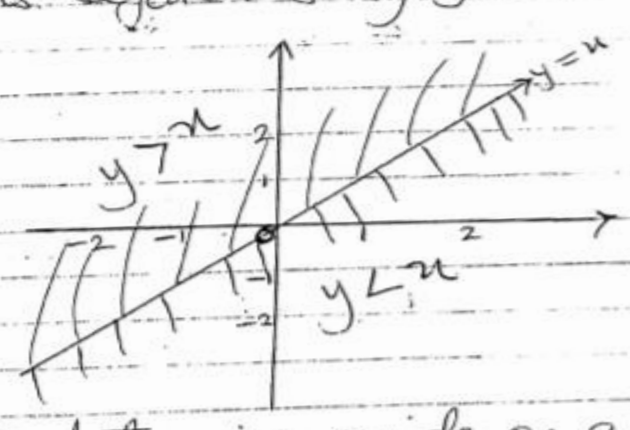
Inequality Graphs

Suppose we ~~wa~~ wish to show which region lies above or below the line $y = x$, first of all, we have to draw the line $y = x$.

We notice that any line in a plane divides the plane into 3 distinct sets of points:
namely, (i) the points on the line itself,
(ii) the points above the line and
(iii) the points below the line.

Half plane

The regions on either side of the line $y = x$ are called half planes. The region above the line $y = x$ is defined as $y > x$, and region below $y = x$ is defined by $y < x$



How to determine points on a straight line

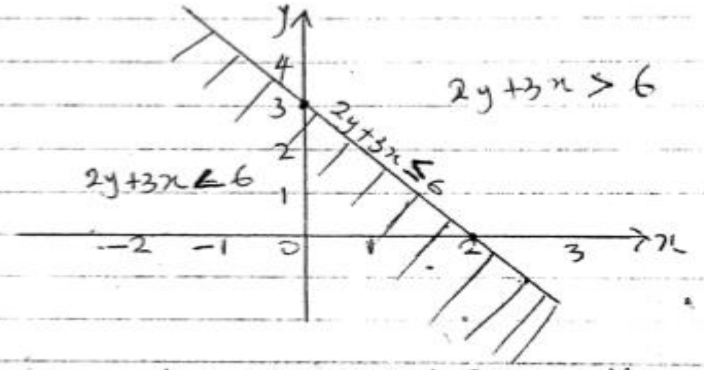
We can draw a straight line very quickly if we know the intercepts of the line with the coordinate axes. Consider the line $2y + 3x = 6$. The convenient approach is to write the equation of the straight line in it intercept form. The line $2y + 3x = 6$, has intercept form given by

$$\frac{y}{3} + \frac{x}{2} = 1.$$

This line makes intercepts $x = 2$ on the x-axis and $y = 3$ on the y-axis.

How to determine the region to be Shaded

Assuming we have selected any three points on the line $2y+3x=6$, how can we Shade the region $2y+3x \leq 6$? The method for deciding which portion of the plane that Should be Shaded is to test the point $(0,0)$ i.e $(x=0, y=0)$ When $x=y=0$, $2y+3x=0<6$. This test, in which the point $(0,0)$, Satisfies the inequality $2y+3x \leq 6$, $(0,0)$ lies below the given line, So we Should shade that Side of the line which contains $(0,0)$.



Suppose we test the point $(1,2)$, then $2y+3x = 2\cdot 2+3\cdot 1 = 7 \not< 6$. That 7 is not less than 6. So we should not Shade the upper region of the line $2y+3x=6$.

Examples

(1) Sketch the region defined by $x+y \leq 4$; $2x-3y \leq 6$; $3x-y \geq -3$ and $x \leq 2$.

Solution

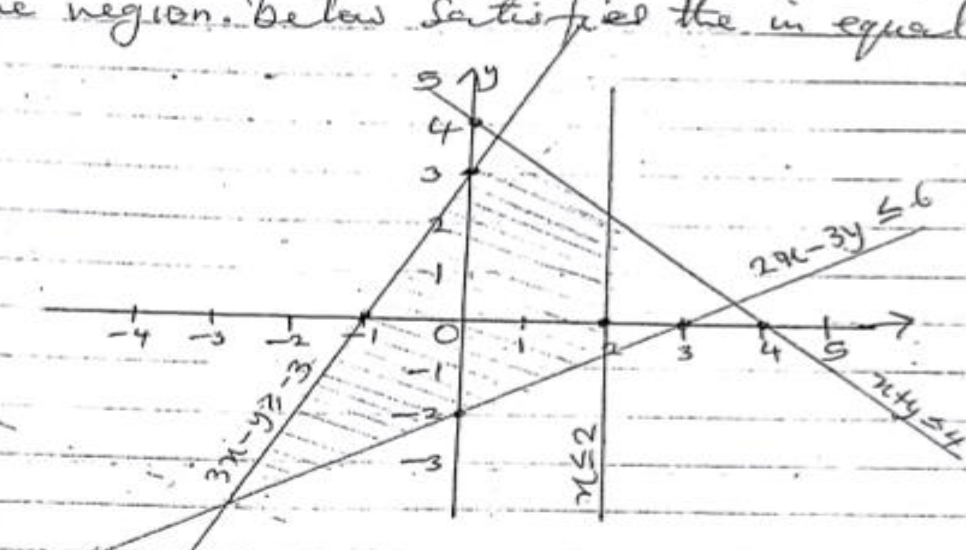
The line $x+y=4$ in its intercept form is

$$\frac{x}{4}+\frac{y}{4}=1.$$

Hence the intercepts on the x and y axes are 4 and 4.

For the line $2x-3y=6$, we have: $\frac{x}{3}-\frac{y}{2}=1$, intercepts are $(3,0)$ and $(0,-2)$

For the line $3x - y = -3$, we have $\frac{x}{-1} + \frac{y}{3} = 1$.
Intercepts are: $(-1,0)$, $(0,3)$.
The region below satisfies the inequalities



2 Shade the region defined simultaneously by $4x+3y-12 \geq 0$, $x+2y-4 \leq 0$ and $x-2y < 4$

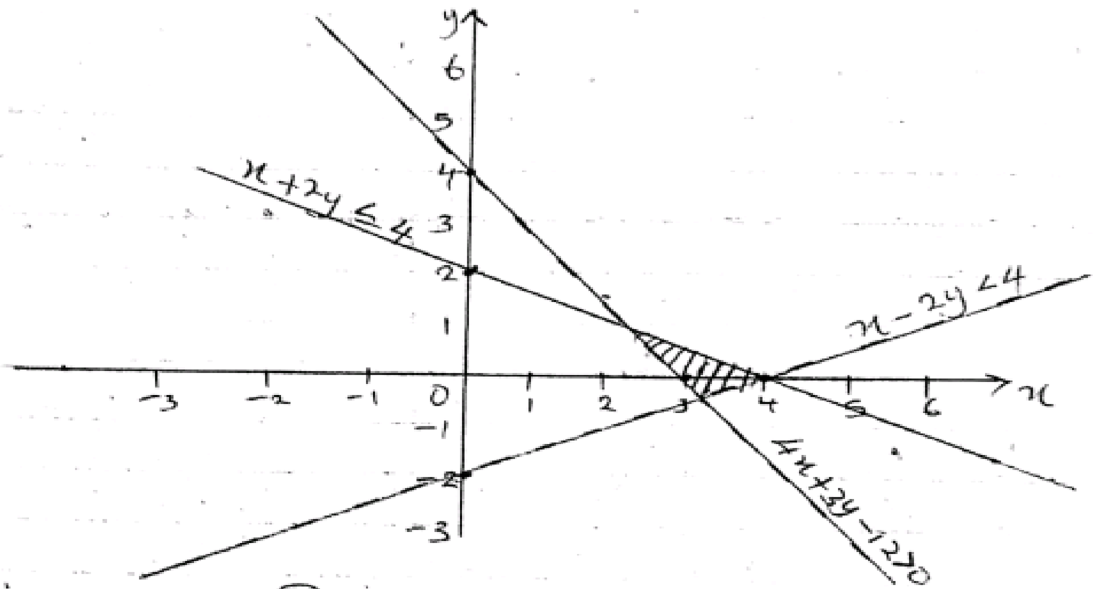
Solution

We first of all locate convenient points of the given. This can be quickly achieved by writing each line in its intercept form

Line	Intercept form	Points on line
$4x+3y-12=0$	$\frac{x}{3}+\frac{y}{4}=1$	$(0,4),(3,0)$
$x+2y-4=0$	$\frac{x}{4}+\frac{y}{2}=1$	$(0,2),(4,0)$
$x-2y=4$	$\frac{x}{4}-\frac{y}{2}=1$	$(0,-2),(4,0)$

With these points, we can draw the graph of the lines in the problem.
Notice that the ~~other two~~ lines $x+2y-4=0$

is not broken like the other two lines. This is to draw a distinction between 'less than or equal to' 'strictly less than' or 'strictly greater than'. The desired region is the shaded portion.



Exercise

In each of question 1-4, Shade the region which is satisfied simultaneously by the lines given:

1. $x > 0,\ y > 0,\ 3x+7y \ge 21,\ 5x+4y \ge 20$
2. $y < 3x,\ 3y > x,\ x+3y < 6$
3. $x+y \le 6,\ x \ge 2,\ y \ge 1$
4. $x > 0,\ y > 0,\ x+3y \le 15$ and $x+y \le 7$.
5. Find out if the region bounded by the lines is closed or open.

(i) $y+3x < 6,\ 2y+x < 3,\ y < x+3$

(ii) $y > 3x,\ y > 2x,\ y > x$.

(iii) $x > 0,\ y > 0,\ 3x+4y \le 12$

(iv) $2x-y \ge 0,\ x+3y \le 14,\ 3x-5y \le 0$

(v) $y-x \le 0,\ x+y \le 2,\ y \ge 2$. (Say whether the region exist or not.)

POLYNOMIALS

Before describing polynomials, let us see what we called a positive integral exponents and terms. For instance: the positive integral n is in the form $x^n = \underbrace{x \cdot x \cdot x \cdot x \cdot \cdots \cdot x}_{n\ factor\ of\ x} \implies x^n$. Example: If $n = 5$ and $x = 2$ then we have $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$. The next thing to consider is "term" a term can be defined as an expression which is either a constant or a product of a constant and a variable. Examples: $81, -6,\ 9x^3,\ x^4y^2,\ -z.\ e.t.c.$

More precisely, an algebraic expression that is made up of one term is called a monomial or term Examples: $81, -6,\ 9x^3,\ x^4y^2,\ -z.\ e \cdot t \cdot c.$

An algebraic expression that is made up of two terms is called a binomial. Examples: $2xy - 3x$, $6x^5 + yz,\ 3a^4 + b^2\ \ e.t.c$ Now instead of inventing more names for this types of expressions with more terms, we give them a general name called polynomial.

Definition: A polynomial is defined as the sum of any two or more number of terms. Examples:
(1) $2x + 7 \implies$ is a polynomial with terms $2x$ & 7, it is a binomial & a polynomial of degree one.
(2) $4a^4 + 6a + 5 \implies$ is a polynomial with terms $4a^4,\ 6a$ & 5, it is a trinomial & a polynomial of degree four. Other examples polynomials include:
(3) $4y^3 - 5y^2 + 7y + 10 \implies$ is a polynomial in as y & is a polynomial of degree three.
(4) $4k^6 - 7k^5 + 6k^4 + 13k - 24 \implies$ is a polynomial in k & is a polynomial of degree six.

In general, if $f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdot \cdot \cdot + k.$ and $a \neq 0$. Then $f(x)$ is said to be a polynomial of degree (order) n in the variable x.

Note: the powers of a polynomial will never be negative, meaning we cannot have $x^{-4} + 6a^{-1}$.

Example: If $f(x) = 3x^2 + x + 5$, find $f(x - 2)$ & compute $f(-1)$ with the answer obtained.

Solution: $f(x - 2) = 3(x - 2)^2 + (x - 2) + 5 \implies 3x^2 - 11x + 15$ and $f(-1) = 29$.

ADDITION AND SUBTRACTION OF POLYNOMIALS

(1) Add the polynomials $4x^2 + 8x + 6$, $2x^3 - 8x^2 + 7$ & $x^4 - 7x^3 + 6x^2 - x + 15$ Solution: $$\begin{array}{r} 0x^4 + 0x^3 + 4x^2 + 8x + 6 \\ 0x^4 + 2x^3 - 8x^2 + 0x + 7 \\ + \quad x^4 - 7x^3 + 6x^2 - x + 15 \\ \hline x^4 - 5x^3 + 2x^2 + 7x + 28 \\ \hline\hline \end{array}$$	(2) Subtract the polynomial $2x^3 + 3x^2 - 2x - 4$ from $6x^4 - 5x^3 + 12x^2 - 8x + 3$ Solution: $$\begin{array}{r} 6x^4 - 5x^3 + 12x^2 - 8x + 3 \\ - \quad 0x^4 + 2x^3 + 3x^2 - 2x - 4 \\ \hline 6x^4 - 7x^3 + 9x^2 - 6x + 7 \\ \hline\hline \end{array}$$

EXERCISE (8)

(1) Add the polynomials $3x^2 - 3x - 1$ and $x^4 - 4x^3 - 7$ with $4x^2 + 8x + 6$?

(2) Add $4x^5 - 7x^4 + 3x^3 - 17x^2 + 15x - 40$ & $10x^5 + 24x^3 - 18x + 7$ with $x^2 + 14$?

(3) Subtract $4x^2 - 5x^4 + 7$ from $10x^2 - 17x + 15$

(3) Subtract $16x^4 + 5x^3 - 12x^2 + 8x - 3$ from $25x^5 - 17x^2 + 14x - 18$

MULTIPLIVATION AND DIVISION OF POLYNOMIALS

(1) Multiply the polynomial $3x^2 - 6x - 4$ by $5x + 2$?

Solution:

$$\begin{array}{rr} & 3x^2 - 6x - 4 \\ \times & 5x + 2 \\ \hline + & 6x^2 - 12x - 8 \\ & 15x^3 - 30x^2 - 20x \\ \hline & 15x^3 - 24x^2 - 32x - 8 \\ \hline\hline \end{array}$$

(2) Find the product of $8x^3 - 5x^2 + 5$ & $5x^3 + x^2 + 8x$

Solution:

$$\begin{array}{rr} & 8x^3 - 5x^2 + 0x + 5 \\ \times & 5x^3 - x^2 + 8x + 0 \\ \hline & 0 - 0 + 0 + 0 \\ + & 64x^4 - 40x^3 + 0x^2 + 40x \\ & -8x^5 + 5x^4 - 0x^3 - 5x^2 \\ & 40x^6 - 25x^5 + 0x^4 + 25x^3 \\ \hline & 40x^6 - 33x^5 + 69x^4 - 15x^3 - 5x^2 + 40x \\ \hline\hline \end{array}$$

(3) Divide the polynomial $5x^3 + 2x^2 - x - 2$ by $x^2 + 2x - 4$?

Solution

$$\begin{array}{r|l} & 5x + 1 \\ \hline x^2 + 2x - 4 & 5x^3 + 2x^2 - x - 2 \\ & -\ (5x^3 + 10x^2 - 20x) \\ \hline & -8x^2 + 19x - 2 \\ & -\ (-8x^2 - 16x + 32) \\ \hline & 35x - 34 \\ \hline\hline \end{array}$$

$\therefore (5x^3 + 2x^2 - x - 2) \div (x^2 + 2x - 4) =$ $(5x + 1)$. And reminder is $35x - 34$.

(4) Divide the polynomial $x^3 - 1$ by $x - 1$.

Solution

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x - 1 & x^3 + 0x^2 + 0x - 1 \\ & -\ (x^3 - x^2) \\ \hline & x^2 + 0x - 1 \\ & -\ (x^2 - x) \\ \hline & x - 1 \\ & -\ (x - 1) \\ \hline & 0 - 0 \\ \hline\hline \end{array}$$

$\therefore (x^3 - 1) \div (x - 1) = (x^2 + x + 1)$.& R= 0

EXERCISE (9)

(1) Multiply the polynomial $x^2 - 7x + 10$ by $x - 6$?

(2) Multiply the polynomial $15x^3 - 12x^2 + 6x - 7$ by $x^2 - 2x + 1$?

(3) Multiply the polynomial $2x^4 - 7x^3 + 4x^2 - 14x + 7$ by $x^3 + 4x^2 - 7x$?

(4) Divide the polynomial $x^2 + 5x + 6$ by $x + 3$?

(5) Divide the polynomial $12x^3 + 23x^2 - 3x - 2$ by $3x - 1$?

(6) Divide the polynomial $2x^3 - 9x^2 - 6x + 40$ by $x^2 - 4$?

REMAINDER AND FACTOR THEOREM

We have seen the operations on polynomial, such as $(+,\ -,\ \times\ \&\ \div)$, the division of a polynomial by another polynomial has given rise to two important theorems, namely:

(1) REMAINDER THEOREM: As the name implies, Remainder means what is left when a number divides another number. For instance

5	Here the number 5 is referred to as the quotient
$4 \,\rfloor\, 21$	Here the number 4 is referred to as the divisor
$-\ \underline{20}$	Here the number 21 is referred to as the dividend
1	Here the number 1 is referred to as the remainder.

Example (1) find the quotient and the remainder of the polynomial $x^3 - 5x + 5 \div x - 2.$

Solution:

$$
\begin{array}{r|l}
 & x^2 + 2x - 1 \\
\hline
x - 2 & x^3 + 0x^2 - 5x + 5 \\
 & -\ \underline{(x^3 - 2x^2)} \\
 & \quad 2x^2 - 5x + 5 \\
 & \quad -\ \underline{(2x^2 - 4x)} \\
 & \qquad -x + 5 \\
 & \qquad -\ \underline{(-x + 2)} \\
 & \qquad\quad 3 \longrightarrow R
\end{array}
$$

So the quotient $= x^2 + 2x - 1$ And reminder is 3.

Hence $x^3 - 5x + 5 \div x - 2$ has the property $(x^2 + 2x - 1)(x - 2) + 3 = x^3 - 5x + 5$

Therefore the division of a polynomial $p(x) \equiv q(x)(Mx + c) + R$. Where $p(x)$ means dividend, $q(x)$ means quotient, $(Mx + c)$ means divisor, and R means remainder.

Note: the R need not to contain x at all, when this happens, we say that R is of degree zero.

Therefore: The remainder theorem states that if a polynomial in x say $p(x)$ is divided by another polynomial $(mx + c)$, then the remainder is equal to $f(a)$, where $f(a)$ is a constant.

Example (2): Find the value of h if $P(y) = y^3 + 2y^2 + hy + 4$ is divided by $y - 2$ & gives a remainder as 14 ?

Solution: If we set $y - 2 = 0$ we get $y = 2$

$\therefore\ y^3 + 2y^2 + hy + 4 = 14$

$\therefore\ y^3 + 2y^2 + hy + 4 = 14$

$\Longrightarrow (2)^3 + 2(2)^2 + h(2) + 4 = 14$

$\Longrightarrow 8 + 8 + 2h + 4 = 14$

$\Longrightarrow 20 + 2h = 14$

$\Longrightarrow 2h = 14 - 20 \quad \& \ h = -3$

$\Longrightarrow h = -3$

$$
\begin{array}{r|l}
 & y^2 + 4y + 5 \\
\hline
y - 2 & y^3 + 2y^2 - 3y + 4 \\
 & -\ (y^3 - 2y^2) \\
\hline
 & 4y^2 - 3y + 4 \\
 & -\ (4y^2 - 8y) \\
\hline
 & 5y + 4 \\
 & -\ (5y - 10) \\
\hline
 & 14 \longrightarrow R
\end{array}
$$

Note that, the reminder of a polynomial can also be found by substituting directly $x = a$ if the divisor is $x - a$.

Example (3): find the remainder of the polynomial $x^3 - 5x + 5 \div x - 2$.

Solution: Let $x - 2 = 0$ we get $x = 2$, then substituting $x = 2$ in the polynomial $x^3 - 5x + 5$
$\Longrightarrow (2)^3 - 5(2) + 5, \quad \Longrightarrow 8 - 10 + 5, \quad \Longrightarrow 13 - 10, \quad \Longrightarrow 3 \quad \therefore$ the remainder is 3.

Further: the alternative method of finding the remainder of a polynomial is by synthetic division

Example (4): find the quotient and remainder of the polynomial $(y^3 + 2y^2 - 3y + 4) \div (y - 2)$

Solution: we set up the synthetic division as follows, so since $y - 2 = 0$ then $y = 2$

$$
\begin{array}{rrrrr|l}
 & 1 & 2 & -3 & 4 & \underline{2} \\
+ & & 2 & 8 & 10 & \\
\hline
 & 1 & 4 & 5 & \boxed{14} &
\end{array}
$$

The no's 1, 2, -3 & 4 are coefficients of the dividend while the no's 1, 4 & 5 are coefficients of the quotient And 14 is the remainder. Thus the quotient is $y^2 + 4y + 5$

Example (5) find the remainder of the polynomial $(x^3 - 4x^2 + 9x - 43) \div (x - 4)$ using synthetic division ?

Solution: we set up the synthetic division as follows, so since $y - 2 = 0$ then $y = 2$

$$
\begin{array}{rrrrr|l}
 & 1 & -4 & 9 & -43 & \underline{4} \\
+ & & 4 & 0 & 36 & \\
\hline
 & 1 & 0 & 9 & \boxed{-7} &
\end{array}
$$

The no's 1, -4, 9 & -43 are coefficients of the dividend while the no's 1, 0, & 9 are coefficients of the quotient And -7 is the remainder. Thus the quotient is $x^2 + 9$

EXERCISE (10)

Find the quotient & remainder of the following polynomials by long division & synthetic method

(1) $(x^3 - 3x^2 + 4x + 2) \div (x + 1)$

(2) $(2x^3 - 3x^2 - 14x - 7) \div (x - 4)$

(3) $(x^2 + 5x + 6) \div (x - 4)$

(4) $(3x^3 + 9x^2 - 11x - 33) \div (x + 3)$

(5) $(4x^2 + 3x^2 + 1) \div (x^2 + 2x - 1)$

(6) $(x^4 - 1) \div (x^2 + 1)$

(2) FACTOR THEOREM: We have seen how we can find the remainder of a polynomial, now if the remainder of a given polynomial gives a zero as a remainder, then we say that the divisor is a factor of the dividend. For instance:

Example (6): find the remainder of the polynomial $(x^3 + 8) \div (x + 2)$.

Solution: Let $x + 2 = 0$ we get $x = -2$, then substituting $x = -2$ in the polynomial $x^3 + 8$ $\Rightarrow (-2)^3 + 8$, $\Rightarrow -8 + 8$, $\Rightarrow 0$, $\therefore$ the remainder is 0. Hence we conclude by saying that, the divisor $x + 2$ is a factor of the dividend $x^3 + 8$.

Example (7): Show that $x - 1$ is a factor of the polynomial $x^3 + 2x^2 - x - 2$, using any method

Solution:

$$\begin{array}{rrrrr|l} & 1 & 2 & -1 & -2 & 2 \\ + & & 1 & 3 & 2 & \\ \hline & 1 & 3 & 2 & \boxed{0} & \end{array}$$

Remainder

Since the remainder is 0, then $(x - 1)$ is a factor of the polynomial $x^3 + 2x^2 - x - 2$

Solution:

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ \hline x - 1 & x^3 + 2x^2 - x - 2 \\ - & (x^3 - x^2) \\ \hline & 3x^2 - x - 2 \\ - & (3x^2 - 3x) \\ \hline & 2x - 2 \\ - & (2x - 2) \\ \hline & 0 - 0 \end{array}$$

$\therefore$ the quotient $= x^2 + 3x + 2$ & the reminder is 0

Example (8): For what value of k is the expression $x^2 + 9x + k$ divisible by $x + 5$?

Solution: Set $x + 5 = 0$, we get $x = -5$ then substituting $x = -5$ in the polynomial $x^2 + 9x + k$ $\Rightarrow (-5)^2 + 9(-5) + k$,

Since $x^2 + 9x + k$ is divisible by $x + 5$, that means it has a remainder of 0.

$\Rightarrow (-5)^2 + 9(-5) + k = 0 \Rightarrow k = 0$,

$\therefore$ the value of k for which $x^2 + 9x + k$ is divisible by $x + 5$ is 20.

Let's Check

Solution

$$\begin{array}{r|l} & x^2 + 3x + 2 \\ \hline x - 1 & x^2 + 9x + 20 \\ - & (x^3 + 5x) \\ \hline & 4x + 20 \\ - & (4x + 20) \\ \hline & 0 - 0 \end{array}$$

So there is no remainder when $k = 0$. $\therefore x + 5$ is a factor of the polynomial $x^2 + 9x + k$.

Example (9): Show without performing long division that $x-1$ is a factor of the polynomial x^3-1 ?

Solution: If $(x-1)$ is a factor, then by the factor theorem $f(1)=0$. Since $f(x)= x^3-1$ then substitute $f(1)= 1^3-1=0$. Therefore $(x-1)$ a factor of the polynomial x^3-1.

Example (10): Show that $2x^3-9x^2-6x+40$ is divisible by $x-4$, and find its other factors if they exist ?

Solution: $\begin{array}{rrrrr\|l} & 2 & -9 & -6 & 40 & 2 \\ + & & 8 & -4 & -40 & \\ \hline & 2 & -1 & -10 & \boxed{0} & \end{array}$ $\therefore$ the quotient will be $2x^2-x-10$	Since $R=0$, then $x-4$ is a factor, now the quotient $2x^2-x-10$ will be factorize to obtain the other factors, $\therefore 2x^2-x-10 \equiv (2x-5)(x+2)$. Hence the other factors are $(2x-5)(x+2)$.

Example (11): If a polynomial $f(x)=(p-1)x^2+px^2+qx+r$ where p, q & r are constants is divided by $(x+2)$ & $(x-1)$, the remainders are -5 and 4 respectively. If $(x-1)$ is a factor of the polynomial $f(x)$, then find the values of the constants p, q & r. Hence, factorize $f(x)$ completely ?

Solution: Setting $x+2=0 \implies x=-2$, then substituting $x=-2$ in the polynomial as (-2) $\implies (p-1)(-2)^3+p(-2)^2+q(-2)+r=-5, \implies -8(p-1)+4p-2q+r=-5 \cdots\cdots (1)$

Again, setting $x-1=0 \implies x=1$, then substituting $x=1$ in the polynomial as $f(1) \implies$ $(p-1)(1)^3+p(1)^2+q(1)+r= 4, \implies (p-1)+p+q+r= 4, \cdots\cdots (2)$

Again, setting $x+1=0 \implies x=-1$, then substituting $x=-1$ in the polynomial as $f(-1)$ $\implies (p-1)(-1)^3+p(-1)^2+q(-1)+r= 0, \implies * (p-1)+p-q+r= 0, \cdots\cdots (3)$

Hence, Solving these equations simultaneously gives $p=3$, $q=0$ & $r=-1$. Therefore the polynomial $f(x)=(p-1)x^2+px^2+qx+r$ becomes $f(x)=2x^3+3x^2-1$.

Now, we need to divide $f(x)$ by $(x+1)$ to obtain the other factors, then using long division and the synthetic method, we have:

<table>
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Solution:

$$\begin{array}{rrrrr|l} & 1 & 2 & -1 & -2 & 2 \\ + & & 1 & 3 & 2 & \\ \hline & 1 & 3 & 2 & \boxed{0} & \end{array}$$

Remainder

Since the remainder is 0, then $(x-1)$ is a factor of the polynomial x^3+2x^2-x-2

</td>
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Solution:

$$\begin{array}{r|l} & 2x^2 + x - 1 \\ \hline x+1 & 2x^3+3x^2+0x-1 \\ & -\ (2x^3-2x^2) \\ \hline & \quad x^2 + 0x - 1 \\ & \quad -\ (x^2 + x) \\ \hline & \qquad -x + 1 \\ & \qquad -\ (-x-1) \\ \hline & \qquad 0 - 0 \end{array}$$

$\therefore\ 2x^3+3x^2-1=(x+1)(2x^2+x-1)=$

$(x+1)\ (2x-1)(x+1).$

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EXERCISE (11)

(1) Show that the polynomial $P(x)=x^4+x^3-7x^2-x+6$ is divisible by $(x-1)$ & $(x-1)$

(2) Show that the polynomial $P(x)=x^3-1^3$ is divisible by $(x-1)$ and find the other factors.

(3) For what value of k the expression $2x^2+kx+2x-15$ is divisible by $x-3$. & show the other factors.

(4) Show without performing long division that $x+1$ is a factor of the polynomial x^3+1.

QUADRATIC EQUATION

There is an equation whose highest power of its variable is always two. It has general representation of the form $ax^2+bx+c=0$, where $a,\ b$ & c are all constants, such that $a\neq 0$. Quadratic equation is usually solved using four (4) methods, they are:

(1) Factorization (2) Completing the square method (3) General method (4) Graphical method

FACTORIZATION METHOD

Example (1) Solve the Quadratic equation $x^2-7x+10=0$ using factorization method ?

Solution: $x^2-7x+10=0 \Rightarrow x^2-5x-2x+10=0 \Rightarrow (x^2-5x)+(-2x+10)=0$

$\Rightarrow -x(x-5)-2(x-5)=0 \Rightarrow (x-5)(x-5)=0 \quad \therefore x=2 \quad or \quad x=5.$

Example (2) Solve the Quadratic equation $2k^2-3k-5=0$ by factorization & obtain it's root.

Solution: $2k^2-3k-5=0 \Rightarrow 2k^2-5k+2k-5=0 \Rightarrow (2k^2-5k)+(2k-5)=0$

$\Rightarrow k(2k-5)+(2k-5)=0 \Rightarrow (2k-5)(k+1)=0 \quad \therefore\ k=\frac{2}{5}\ or\ k=-1$ are the roots.

Example (3) Find the value of the unknown x in the given Quadratic equation $2x^2+6x=0$?

Solution: $2x^2+6x=0 \Rightarrow 2x(x+3)=0 \Rightarrow 2x=0\ or\ (x+5)=0\ \therefore x=0\ or\ x=-3$

Example (4) Solve the Quadratic equation $3(3^{2x}) - 10 \cdot 3^x + 3 = 0$ using factorization method.

Solution: Let $y = 3^x \implies 3(3^{2x}) - 10 \cdot 3^x - 3 = 0 \equiv 3y^2 - 10y + 3 = 0$

$3y^2 - 10y + 3 = 0 \quad \implies 3y^2 - 9y - y + 3 = 0 \quad \implies (3y^2 - 9y) + (-y + 3) = 0$

$\implies 3y(y-3) - (y-3) = 0 \quad \implies (3y-1)(y-3) = 0 \quad \implies y = \frac{1}{3} \quad or \quad y = 3$

$\implies \frac{1}{3} = 3^x$ or $3 = 3^x$ $\quad \therefore x = -1 \quad or \quad x = 1.$

Example (5) Find the value of the unknown x in the given equation $(x+1)^2 = 9/25$?

Solution: $(x+1)^2 = 9/25 \quad \implies (x+1)(x+1) = 9/25 \quad \implies x^2 + 2x + 1 = 9/25$

$\implies 25x^2 + 50x + 16 = 0 \quad \implies (25x^2 + 40x) + (10x + 16) = 0 \quad \implies 5x(5x+8) + 2(2x+8) = 0 \quad \implies (5x+2)(5x+8) = 0 \quad \therefore x = -2/5 \quad or \quad x = -8/5.$

Note: You can only solve use factorization method to solve a given problem if the discriminant of the Quadratic equation is a perfect square.

METHOD OF COMPLETING THE SQUARE

There are some quadratic equations that cannot be solve using factorization method, then we need a general method, so one such method is by completing the square method, therefore this method involves making the expression of the equation to a perfect square which can be factorized to solve.

Example (6) Solve the quadratic equation $3x^2 - 5x - 7 = 0$ using completing the square method.

Solution: Dividing through by 3 gives $x^2 - \frac{5}{3}x = \frac{7}{3} \implies x^2 - \frac{5}{3}x + \left[\frac{-1}{2} \times \frac{5}{3}\right]^2 = \frac{7}{3} + \left[\frac{-1}{2} \times \frac{5}{3}\right]^2$

$\implies x^2 - \frac{5}{3}x + \left[\frac{-5}{6}\right]^2 = \frac{7}{3} + \left[\frac{-5}{6}\right]^2 \quad \implies \left[x - \frac{5}{6}\right]^2 = \frac{109}{36} \quad \implies \sqrt{\left[x - \frac{5}{6}\right]^2} = \pm\sqrt{\frac{109}{36}}$

$\implies x - \frac{5}{6} = \pm \frac{\sqrt{109}}{6} \quad \implies x = \frac{5}{6} \pm \frac{\sqrt{109}}{6} \quad \therefore x_1 \approx 2.6 \quad or \quad x_2 \approx -0 \cdot 9.$

Example (7) Solve the quadratic equation $2t^2 + 16t - 40 = 0$ by completing the square method.

Solution: Dividing through by 2 gives $t^2 + \frac{16}{2}t = \frac{40}{2} \implies t^2 + 8t + \left[\frac{1}{2} \times 8\right]^2 = 20 + \left[\frac{1}{2} \times 8\right]^2$

$\implies t^2 + 8t + [\,4\,]^2 = 20 + [\,4\,]^2 \quad \implies [\,t+4\,]^2 = 36 \quad \implies \sqrt{[\,t+4\,]^2} = \pm\sqrt{36}$

$\implies t + 4 = \pm\sqrt{36} \quad \implies t = -4 \pm \sqrt{36} \quad \therefore t_1 = 2 \quad or \quad t_2 = -10.$

Example (8) Find the value of x in the equation $(x+1)^2 = \frac{9}{25}$ using completing the square

Solution: $(x+1)^2 = \frac{9}{25}$ take the square root of both sides $\implies \sqrt{[\,x+1\,]^2} = \pm\sqrt{\frac{9}{25}}$

$\implies x + 1 = \pm \frac{3}{5} \quad \implies x = -1 \pm \frac{3}{5} \quad \therefore x_1 = -\frac{2}{5} \quad or \quad x_2 = -\frac{8}{5}.$

Note: First make sure the coefficient of x^2 is one, secondly transfer the constant to the RHS. Next take half coefficient of x, square it and add to both sides.

QUADRATIC FORMULAR (GENERAL METHOD)

This formula is also used when a quadratic equation cannot be solve using factorization method, the formula is derived through the method completing the square using general representation of the equation $ax^2 + bx + c = 0$ and arrived at $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ .... (★)

Example (9) Use completing the square method to drive the quadratic formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ using the equation $ax^2 + bx - c = 0$.

Solution: Dividing through by a gives $x^2 + \frac{b}{a}x = \frac{-c}{a} \implies x^2 + \frac{b}{a}x + \left[\frac{1}{2} \times \frac{b}{a}\right]^2 = \frac{-c}{a} + \left[\frac{1}{2} \times \frac{b}{a}\right]^2$

$\implies x^2 + \frac{b}{a}x + \left[\frac{b}{2a}\right]^2 = \frac{-c}{a} + \left[\frac{b}{2a}\right]^2 \implies \left[x + \frac{b}{2a}\right]^2 = \frac{-c}{a} + \frac{b}{2a} \implies \left[x + \frac{b}{2a}\right]^2 = \frac{-4ac+b^2}{4a^2}$

$\implies \sqrt{\left[x + \frac{b}{2a}\right]^2} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} \implies x + \frac{b}{2a} = \pm \frac{\sqrt{b^2-4ac}}{2a} \implies x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

$\therefore \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ .... (★) this great equation is called the general formula.

Example (10) Solve the Quadratic equation $x^2 - 7x + 10 = 0$ using the general formula ? Solution: Here our $a = 1$, $b = -7$ & $c = 10$ then substituting into the general formula we have

$\implies x = \frac{-(-7) \pm \sqrt{(-7)^2-4(1)(10)}}{2(1)} \implies x = \frac{7 \pm \sqrt{49-40}}{2} \implies x = \frac{7 \pm \sqrt{9}}{2} \quad \therefore x_1 = 5 \quad or \quad x_2 = 2$

Example (11) Solve the Quadratic equation $2y^2 - 3y - 5 = 0$ using the general formula ? Solution: Here our $a = 2$, $b = -3$ & $c = -5$ then substituting into the general formula we have

$\implies x = \frac{-(-3) \pm \sqrt{(-3)^2-4(2)(-5)}}{2(2)} \implies x = \frac{3 \pm \sqrt{9+40}}{4} \implies x = \frac{3 \pm 7}{4} \quad \therefore x_1 = \frac{5}{2} \quad or \quad x_2 = -1.$

Note: In solving quadratic equation, the discriminant always determines the type of roots: that is

(a) If $b^2 - 4ac > 0$, then the roots are real and distinct.

(b) If $b^2 - 4ac = 0$, then the roots are real and equal.

(c) If $b^2 - 4ac < 0$, then there are no roots.

GRAPHICAL METHOD OF SOLVING QUADRATIC EQUATION

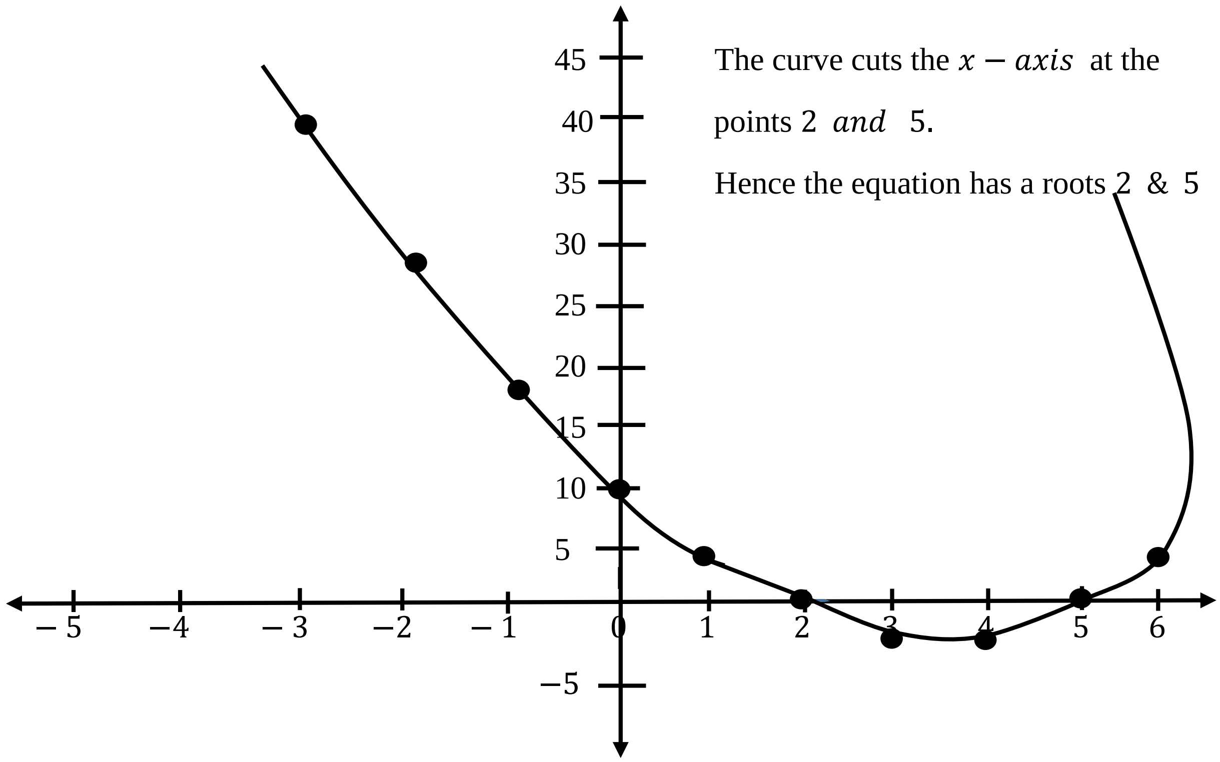
Consider the general quadratic equation $ax^2 + bx + c = 0$. Then the relation $y = ax^2 + bx + c$ is called the quadratic function of x. This method of solving quadratic equation is by plotting a graph of y against x.

The graph is a curve and is called a **parabola** while the resulting curve gives the root of the equation at where they intersect the $x - axis$. And the graph is always either in $\cup$ or $\cap$.

Example (12) Solve the Quadratic equation $x^2 - 7x + 10 = 0$ using the graphical method ?

Solution: draw a table containing set of points from -3 $-$ 3 to plot the graph as shown below

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
$-7x$	21	14	7	0	-7	-14	-21	-28	-35	-42
10	10	10	10	10	10	10	10	10	10	10
y	40	28	18	10	4	0	-2	-2	0	4



The curve cuts the $x - axis$ at the points 2 and 5.

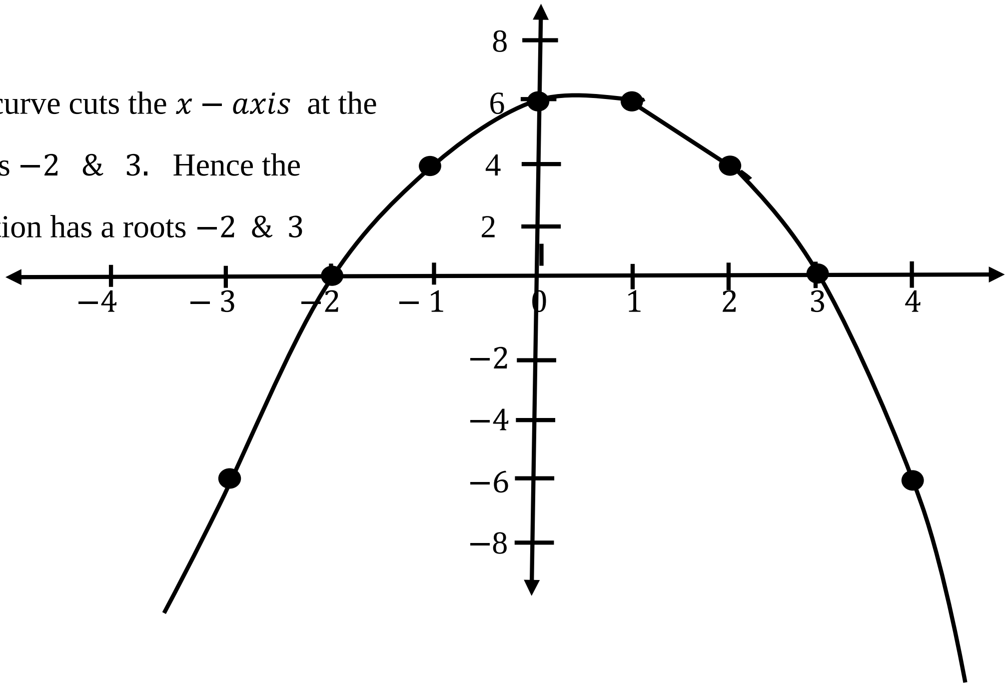
Hence the equation has a roots 2 & 5

Example (13) Solve the Quadratic equation $6 + x - x^2 = 0$ for values of x from -3 to 3 ?

Solution: draw a table containing set of points from -3 $-$ 3 to plot the graph as shown below

x	-3	-2	-1	0	1	2	3
6	6	6	6	6	6	6	6
x	-3	-2	-1	0	1	2	3
$-x^2$	-9	-4	-1	0	-1	-4	-9
y	-6	0	4	6	6	4	0

The curve cuts the $x - axis$ at the points -2 & 3. Hence the equation has a roots -2 & 3



EXERCISE (12)

(1) Find the value of the unknown in the following equations below by using factorization method

(a) $x^2 + 7x + 12 = 0$ (b) $3x^2 + 5x - 2 = 0$ (c) $(x + 1)^2 = 25$ (d) $4x^2 + 14x = 0$

(2) Find the value of the unknowns by using completing the square method for each of the quadratic equations below: (a) $x^2 - 10x + 15 = 0$ (b) $3x^2 = 31x + 22$

(c) $3(3x^2 + 1) = 12x$ (d) $x^2 = 2(1 - x)$ (e) $\frac{b+4}{b-a} = b$ (f) $\frac{a-3}{a-4} + \frac{a+3}{a+4} = 0$

(3) Draw a graph to find the roots of the quadratic equations below by using graphical method:

(a) $x^2 - 2x - 3 = 0$ (b) $4x^2 * 20x + 25 = 0$ (a) $x^2 6x + 3 = 0$ (b) $5x^2 + 2x + 3 = 0$

(4) Use general formula to solve all the above questions and compare the answers obtained ?

FORMATATION OF QUADRATIC EQUATION FROM A GIVEN ROOTS

Sometimes the roots of a particular equation may be given, & it's equation will be required. So to achieve this assertion, we use this formula $x^2 -$ (sum of the roots)$x +$ (product of the roots) $= 0$.

Example (1): Find the quadratic equation whose roots are 2 and 5 ?

Solution: Using the above formula $x^2 -$ (sum of the roots)x $+$ (product of the roots) $= 0$.

$\Rightarrow x^2 - (2 + 5)x + (2 \times 5) = 0$ $\Rightarrow x^2 - (7)x + (10) = 0$ $\therefore$ the required equation is $x^2 - 7x + 10 = 0$.

Aliter: Let $x = 2$ & $x = 5$ $\Rightarrow x - 2 = 0$ & $x - 5 = 0$ $\Rightarrow (x-2)(x-5) = 0$ $\Rightarrow x^2 - (5)x - 2x + (10) = 0$ $\therefore$ the required equation is $x^2 - 7x + 10 = 0$

Example (2): Find the quadratic equation whose roots are -4 *and* $-2/3$?

Solution: Using the above formula $x^2 -$ (sum of the roots)x + (product of the roots) $= 0$.

$\Rightarrow \quad x^2 - \left(-4 - \frac{2}{3}\right)x + \left(-4 \times -\frac{2}{3}\right) = 0 \qquad \Rightarrow \quad x^2 - \left(-\frac{14}{3}\right)x + \left(\frac{8}{3}\right) = 0$

$\Rightarrow 3x^2 - 3\left(-\frac{14}{3}\right)x + 3\left(-\frac{8}{3}\right) = 0$ $\therefore$ the required equation is $3x^2 + 14x + 8 = 0$

Aliter: Let $x = -4$ & $x = -\frac{2}{3}$ $\Rightarrow x + 4 = 0$ & $x + \frac{2}{3} = 0$ $\Rightarrow (x+4)\left(x + \frac{2}{3}\right) = 0 \Rightarrow$ $x^2 + \left(\frac{2}{3}\right)x + 4x + \left(\frac{8}{3}\right) = 0 \Rightarrow 3x^2 + 3\left(\frac{2}{3}\right)x + 3(4x) + 3\left(\frac{8}{3}\right) = 0 \Rightarrow 3x^2 + 14x + 8 = 0$

EXERCISE (13)

Form a quadratic equations using the roots (a) -3 & 5 (b) -2 & 7 (c) 0 & 4 (d) $-\frac{2}{5}$ & $-\frac{8}{5}$

SUM & PRODUCT OF A ROOTS AND THEIR SYMMETRIC IDENTITIES

Sometimes we may be required to find the sum and product of the roots of a given quadratic equations without necessarily solving the equations. Now if α & β are the roots of the equation $ax^2 + bx + c = 0$, $\alpha = \frac{-b + \sqrt{D}}{2a}$ and $\beta = \frac{-b - \sqrt{D}}{2a}$ Where $D = b^2 - 4ac$ & is called the discriminant of the equation. So we have the following arguments:

(a) $\alpha + \beta = \frac{-b + \sqrt{D}}{2a} + \frac{-b - \sqrt{D}}{2a} = -\frac{2b}{2a} \Rightarrow -\frac{b}{a}$ So $\alpha + \beta = -\frac{b}{a}$ .... (i)

(b) $\alpha \cdot \beta = \left(\frac{-b + \sqrt{D}}{2a}\right)\left(\frac{-b - \sqrt{D}}{2a}\right) = -\frac{b^2 + b\sqrt{D} - b\sqrt{D} + D}{4a^2} = -\frac{b^2 - D}{4a^2} = -\frac{b^2 - (b^2 - 4ac)}{4a^2}$ So $\alpha \cdot \beta = \frac{c}{a}$ ···(ii)

(c) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

(d) $\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$

(e) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

(f) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

Example (1): Find the sum and product of the roots of the quadratic equation $3x^2 - 4x - 1 = 0$.

Solution: $\frac{3x^2}{3} - \frac{4x}{3} - \frac{1}{3} = \frac{0}{3} = x^2 - \frac{4x}{3} - \frac{1}{3} = 0$ So here $a = 3$, $b = -4$ *and* $c = -1$

$\alpha + \beta = -\frac{b}{a} \Rightarrow -\left(-\frac{4}{3}\right) = \frac{4}{3}$ and $\alpha \cdot \beta = \frac{c}{a} \Rightarrow -\frac{1}{3}$.

Example (2) : If α & β are the roots of the quadratic equation $2x^2 - x - 2 = 0$, then find the values of the following (i) $\alpha^2 + \beta^2$ (ii) $\alpha^2 - \beta^2$ (iii) $\alpha^3 + \beta^3$ (iv) $\alpha^3 - \beta^3$ (v) $\alpha - \beta$ (vi) $\frac{1}{\alpha} + \frac{1}{\beta}$ (vii) $\frac{1}{\alpha} - \frac{1}{\beta}$.

Solution: (i) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ Here each of the new function must be expressed interms of $\alpha+\beta$ and $\alpha\cdot\beta$ Therefore $\alpha^2+\beta^2 = \left(\frac{1}{2}\right)^2 - 2(-1) \implies \frac{1}{4}+2 \implies \frac{9}{4}$

(ii) $\alpha^2-\beta^2 = (\alpha-\beta)(\alpha+\beta) \implies$ Different of two squares, but $\alpha-\beta$ cannot be solve directly, we can use the fact that $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta \implies \left(\frac{1}{2}\right)^2 - 4(-1) \implies \frac{1}{4}+4$ $\implies \frac{17}{4}$. Then $(\alpha-\beta) = \sqrt{\frac{17}{4}} \implies \frac{\sqrt{17}}{2}$. Therefore $(\alpha-\beta)(\alpha+\beta) \implies \left(\frac{\sqrt{17}}{2}\right)\left(\frac{1}{2}\right) \implies \frac{\sqrt{17}}{4}$.

(iii) $\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2) \implies$ By factorization, $\implies \frac{1}{2}\left[\frac{9}{4}-(-1)\right] \implies \frac{13}{8}$.

(iv) $\alpha^3-\beta^3 = (\alpha-\beta)[(\alpha+\beta)^2-\alpha\beta)] \implies \frac{\sqrt{17}}{2}\left[\left(\frac{1}{2}\right)^2-(-1)\right] \implies \frac{\sqrt{17}}{2}\left[\frac{5}{4}\right] \implies \frac{5\sqrt{17}}{8}$.

(v) Solved as $\frac{\sqrt{17}}{2}$ (vi) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\cdot\beta} = \frac{1/2}{-1} \implies -\frac{1}{2}$ (vii) $\frac{1}{\alpha}-\frac{1}{\beta} = \frac{\alpha-\beta}{\alpha\cdot\beta} = \frac{\sqrt{17}/2}{-1} \implies -\frac{\sqrt{17}}{2}$

(3) The quadratic equation $2x^2-3x+6=0$ has two roots α & β, Obtain a quadratic equation in x which has α^3 & β^3 as its roots.

Solution: From $2x^2-3x+6=0$ we have $a=2$, $b=-3$ and $c=6$. So $\alpha+\beta = -\frac{b}{a}$ $\implies \frac{3}{2}$ and $\alpha\beta = \frac{6}{2} \implies 3$, then $\alpha^3+\beta^3 = (\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$ $\implies \alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) \implies \left(\frac{3}{2}\right)^3 - 3(3)\left(\frac{3}{2}\right) \implies -\frac{81}{8}$. And $\alpha^3\beta^3 = (\alpha\beta)^3 = (3)^3 = 27$. So the required equation is $x^2+\frac{81x}{8}+27=0 \implies 8x^2+81x+216=0$

EXERCISE (14)

(1) If α & β are the roots of the quadratic equation $3x^2+5x-2=0$, then find the values of
(a) $\alpha+\beta$ (b) $\alpha^2+\beta^2$ (c) $\alpha^2-\beta^2$ (d) $\alpha^3+\beta^3$ (e) $\alpha^3-\beta^3$
(f) $\alpha-\beta$ (g) $\frac{1}{\alpha}+\frac{1}{\beta}$ (8) $\frac{1}{\alpha}-\frac{1}{\beta}$ (g) $\left(\frac{1}{\alpha}\right)^2+\left(\frac{1}{\beta}\right)^2$ (g) $\left(\frac{1}{\alpha}\right)^2-\left(\frac{1}{\beta}\right)^2$

(2) If α & β are the roots of the quadratic equation $18x^2=9x+5$, then find the values of
(a) $\alpha+\beta$ (b) $\alpha^2+\beta^2$ (c) $\alpha^2-\beta^2$ (d) $\alpha^3+\beta^3$ (e) $\alpha^3-\beta^3$
(f) $\alpha-\beta$ (g) $\frac{1}{\alpha}+\frac{1}{\beta}$ (8) $\frac{1}{\alpha}-\frac{1}{\beta}$ (g) $\left(\frac{1}{\alpha}\right)^2+\left(\frac{1}{\beta}\right)^2$ (g) $\left(\frac{1}{\alpha}\right)^2-\left(\frac{1}{\beta}\right)^2$

RATIONAL FUNCTION

A rational function is any function which can be defined by a rational fraction, it is an algebraic fraction which has both numerator and denominator are polynomials .Mathematically, a function $f(x)$ is called a rational function if and only if it can be written in the form $f(x)$ $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$.

Example (1) $y = \frac{x^3-2x}{2(x^2-5)}$ is rational function of degree 3.

Example (2) $y = \frac{x^2-3x-2}{x^2-4}$ is rational function of degree 2.

PARTIAL FRACTION DECOMPOSITION

In algebra, the partial fraction decomposition (or expansion) of a rational function is the operation that consists of expressing the function as a sum of polynomials and one several function with a simpler denominator.

For instance: we know that $\frac{1}{x+3} + \frac{1}{x-1}$ can be expressed as a single fraction $\frac{3x+5}{(x+3)(x-1)}$, but if we were given $\frac{3x+5}{(x+3)(x-1)}$ how would we get back to $\frac{1}{x+3} + \frac{1}{x-1}$? Now, this reverse process is called expressing the $\frac{3x+5}{(x+3)(x-1)}$ in partial fractions (or resolution of the fraction) in to partial fraction.

The method used is to assume that $\frac{3x+5}{(x+3)(x-1)} \equiv \frac{A}{x+3} + \frac{B}{x-1}$ the denominators of the algebraic fraction encountered will be one of these three basic type in the table.

Type	Denominator with fraction	Example	Expression used
1	Denominator with linear factor	$\frac{5}{(x+3)(x-1)}$	$\frac{A}{x+3} + \frac{B}{x-1}$
2	Denominator with quadratic factor	$\frac{3x+5}{(x-1)(x^2+4)}$	$\frac{A}{x-1} + \frac{Bx+C}{x^2+4}$
3	Denominator with repeated factor	$\frac{3x+5}{(x-1)(x+3)^2}$	$\frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

Example (1): Express the rational function $\frac{1}{(x-1)(x+3)}$ into a partial fraction ?

Solution: $\frac{1}{(x-1)(x+3)} \equiv \frac{A}{x-1} + \frac{B}{x+3}$ multiplying the numerator of both sides by $(x-1)(x+3)$ gives $1 \equiv A(x+3) + B(x-1)$ $\cdots\cdots\cdots\cdots\cdots\cdots\cdots(*)$

Now let $x-1=0$ and $x+3=0$ yields $x=1$ and $x=-3$ substituting $x=1$ into $\cdots\cdots\cdots\cdots\cdots\cdots\cdots(*)$ gives $A=\frac{1}{4}$ and $x=-3$ gives $B=-\frac{1}{4}$.

Hence $\frac{1}{(x-1)(x+3)} \equiv \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+3}\right)$ as the required partial fraction.

Example (2): Resolve the rational function $f(x) = \frac{2x+3}{(x^2-1)(x-2)}$ into a partial fraction ?

Solution: $\frac{2x+3}{(x^2-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2}$ multiplying the numerator of both sides by $(x^2-1)(x-2)$ gives$2x+3 \equiv A(x-1)(x-2) + B(x+1)(x-2) + C(x-1)(x-1) \cdots\cdots\cdots\cdots\cdots\cdots\cdots(\ast)$ Now let $x-1=0$, $x+1=0$ and $x-2=0$ yields $x=1$, $x=-1$ and $x=2$ substituting $x=1$ into $\cdots\cdots\cdots\cdots\cdots\cdots\cdots(\ast)$ gives $B=-\frac{5}{2}$ while $x=-1$ gives $A=\frac{1}{6}$ and $x=2$ gives $B=\frac{7}{3}$. Hence $\frac{2x+3}{(x^2-1)(x-2)} \equiv \frac{1}{6(x-1)} - \frac{5}{2(x+1)} + \frac{7}{3(x-2)}$ as the required partial fraction.

Example (3): Express the rational function $\frac{2x+3}{(x+2)(x^2+1)}$ into a sum of partial fraction ?

Solution: $\frac{2x+3}{(x+2)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ multiplying the numerator of both sides by $(x+2)(x^2+1)$ gives $3x+1 \equiv A(x^2+1) + (Bx+C)(x+2) \cdots\cdots\cdots\cdots(\ast)$ Now letting $x+2=0$ and $x=0$ yields $x=-2$ and $x=0$ substituting $x=-2$ into $\cdots\cdots\cdots\cdots\cdots\cdots\cdots(\ast)$ gives $A=-1$ and $x=0$ gives $C=1$. And equating the coefficient of x^2 gives $0x^2 = Ax^2 + Bx^2 \implies 0 = A+B \implies 0=-1+B$. Hence $\frac{2x+3}{(x+2)(x^2+1)} \equiv \frac{-1}{x+2} + \frac{1}{x^2+1}$ as the required partial fraction.

(4) Solve the given rational function $f(x) = \frac{3x^2+x+9}{(x+3)(x^2+x+5)}$ as a sum in a partial fraction ?

Solution: $\frac{3x^2+x+9}{(x+3)(x^2+x+5)} \equiv \frac{A}{x+3} + \frac{Bx^2+Cx+D}{x^2+x+5}$ multiplying the numerator of both sides by $(x+3)(x^2+x+5)$ gives $3x^2+x+9 \equiv A(x^2+x+5) + (Bx+Cx+D)(x+3) \cdots\cdots\cdots\cdots(\ast)$ Now letting $x+3=0$ yields $x=-3$ substituting $x=-3$ into $\cdots\cdots\cdots\cdots\cdots\cdots\cdots(\ast)$ gives $A=3$.

Then equating the coefficient of x gives $1 = A+D \implies D = 1-3$ and $D=-2$ again equating the coefficient of x^2 gives $3 = A+C \implies C = 3-3$ and $C=0$ Lastly, equating the coefficient of x^3 gives $0x^3 = 0x^3 + Bx^2 \implies B = 0+0$ & $B=0$

Hence $\frac{3x^2+x+9}{(x+3)(x^2+x+5)} \equiv \frac{3}{x+3} - \frac{2}{x^2+x+5}$ as the required partial fraction.

(5) Express the rational function $f(x) = \frac{x+4}{(x+1)(x-2)^2}$ as a sum in a partial fraction ?

Solution: $\frac{x+4}{(x+1)(x-2)^2} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$ multiplying the numerator of both sides by $(x-2)(x-2)^2$ gives $x + 4 \equiv A(x-2)^2 + B(x+1)(x-2) + C(x+1) \cdots (*)$ Now letting $x+1=0$ and $x-2=0$ yields $x=-1$ and $x=2$ substituting $x=-1$ into $\cdots\cdots\cdots\cdots\cdots\cdots\cdots (*)$ gives $A = 1/3$ and $C = 2$

And equating the coefficient of x^3 gives $0x^2 = Ax^2 + Bx^2 \implies 0 = A + B$ & $B = -1/3$

Hence $\frac{x+4}{(x+1)(x-2)^2} \equiv \frac{1}{3(x+1)} - \frac{1}{3(x-2)} + \frac{2}{(x-2)^2}$ as the required partial fraction.

(6) Express the rational function $f(x) = \frac{5x^2-6x-21}{(x-4)^2(2x-3)}$ as a sum in a partial fraction ?

Solution: $\frac{5x^2-6x-21}{(x-4)^2(2x-3)} \equiv \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{(2x-3)}$ multiplying the numerator of both sides by $(x-4)^2(2x-3)$ gives $5x^2 - 6x - 21 \equiv A(x-4)(2x-3) + B(2x-3) + C(x-4)^2 \cdots\cdots\cdots\cdots (*)$ Now letting $x-4=0$ and $2x-3=0$ yields $x=4$ and $x=3/2$ substituting $x=4$ and $x=3/2$ into $\cdots\cdots\cdots\cdots\cdots\cdots (*)$ gives $B=7$ and $C=-3$

And equating the coefficient of x^2 gives $5x^2 = 2Ax^2 + Cx^2 \implies 5 = 2A + C$ & $A = 4$

Hence $\frac{5x^2-6x-21}{(x-4)^2(2x-3)} \equiv \frac{4}{x-4} + \frac{7}{(x-4)^2} - \frac{3}{(2x-3)}$ as the required partial fraction.

EXERCISE (15)

(1) Resolve the following given rational functions as a sum in a partial fraction decomposition ?

$(a)\ \frac{1+x}{(1-x)(2+x)}$ $(b)\ \frac{30-6x}{(x-7)(4-x)}$ $(c)\ \frac{1}{x^3-9x}$ $(d)\ \frac{2x+4}{(x-1)(x+3)}$ $(e)\ \frac{2x^2+17x+21}{(x+2)(x+3)(x-3)}$

(2) Resolve the following given rational functions as a sum in a partial fraction decomposition ?

$(a) \frac{8x-1}{(x-2)(x^2+1)}$ $(b) \frac{5x^2-2x-1}{(x+1)(x^2+1)}$ $(c) \frac{2x^2+17x+21}{(x+2)(x^2+2)}$ $(d) \frac{x+1}{(x-1)(x^2+1)}$ $(e) \frac{21x-7}{(x-2)(x+3)(x^2+x+1)}$

(3) Express each of the following rational functions as a sum into a partial fraction decomposition

$(a)\ \frac{x^2+6x+9}{(x-3)^2(x+5)}$ $(b)\ \frac{7x^2-10x+10}{(x-1)^3}$ $(c)\ \frac{3}{(x+1)(x-2)^2}$ $(d)\ \frac{x-3-2x^2}{x^2(x-1)}$ $(e)\ \frac{3x^2+2}{x(x-1)^2}$

LINEAR EQUATIONS AND CONSISTENCY

SYSTEM OF LINEAR EQUATIONS ONE IN VARIABLE.

Definition; An equation is a mathematical statement that relates two algebraic expressions are equal in value. An equation whose highest power of the unknown is 1 is called a linear equation. Forinstance: $2x + 3 = 7$ and $x + y = 5$ e.t.c. To solve an equation generally means to find the value of the unknown(s) that satisfies the equations.

Examples: solve the following system of linear equations in one variable ?

(1) $2x + 3 = 7$. Solution: $2x + 3 = 7$ collecting like terms gives $2x = 4 \implies 2x = 4 \therefore x = 2$.

(2) $5y = 8 + 3y$. Solution: $5y = 8 + 3y$ collecting like terms gives $2y = 8 \therefore y = 4$.

(3) $3(4y - 7) - 4(4y - 1) = 0$. Solution: simplifying $3(4y - 7) - 4(4y - 1) = 0$ gives $12y - 21 - 16y + 4 = 0 \implies -4y - 17 \therefore y = \frac{-17}{4} \quad or \quad -4\frac{1}{4}$

(4) $\frac{3}{4}(x - 2) + \frac{1}{3}(2x + 5) = 4$. Solution: simplifying $\frac{3}{4}(x - 2) + \frac{1}{3}(2x + 5) = 4$ gives $\frac{3}{4}(x - 2) \times 12 + \frac{1}{3}(2x + 5) \times 12 = 4 \times 12 \implies 17x = 46 \therefore x = \frac{46}{17} \ or \ 2\frac{12}{17}$

SYSTEM OF LINEAR EQUATIONS IN TWO VARIABLES (SIMULTANEOUS EQNS)

In order to solve a system of equations in two unknowns (variables) at the same time (Simultaneously) then the values of the unknowns must satisfy the equations. For instance:

$2x + 3y = -2 \quad ------ \quad (1)$

$3x + 4y = -6 \quad ------ \quad (2)$

These types of equations can be solved basically by three methods namely: (i) by substitution method (ii) by elimination method (iii) by graphical method.

Examples (1): solve the following System of Linear Equations using the above methods stated:
$x + y = 5$ ----------- (1) and $x - y = 3$ ----------- (2)

Solution: Using substitution method, $x + y = 5$ ----- (1) becomes $x = 5 - y$ substituting the x into equation (2) gives $5 - y - y = 3$ ------ (2) $\Rightarrow -2y = 3 - 5$ ----- (2) $\Rightarrow y = 1$. Now substituting $y = 1$ into equation (1) gives $x + (1) = 5$ ------ (1) $\Rightarrow x = 5 - 1 \Rightarrow x = 4$. Thus, we have $x = 4$ & $y = 1$ as the only solution for the given system.

Aliter: Using elimination method

$$\begin{array}{rl} x + y = 5 & \text{-----------} \ (1) \\ - \quad \underline{x - y = 3} & \text{-----------} \ (2) \\ 0x + 2y = 2 & \therefore \ y = 1 \end{array}$$

Putting $y = 1$ into equation (1) gives $x + (1) = 5 \Rightarrow x = 4$. Thus, we have $x = 4$ & $y = 1$.

Aliter: To solve this type of problem graphically, we first draw the graph of two equations simultaneously to get two different straight lines intersecting each other at a common point, and this common point gives the solution of the simultaneous equations. Now, we will apply the trial and error method to find some pairs of values of (x, y) which will satisfy the given equations.

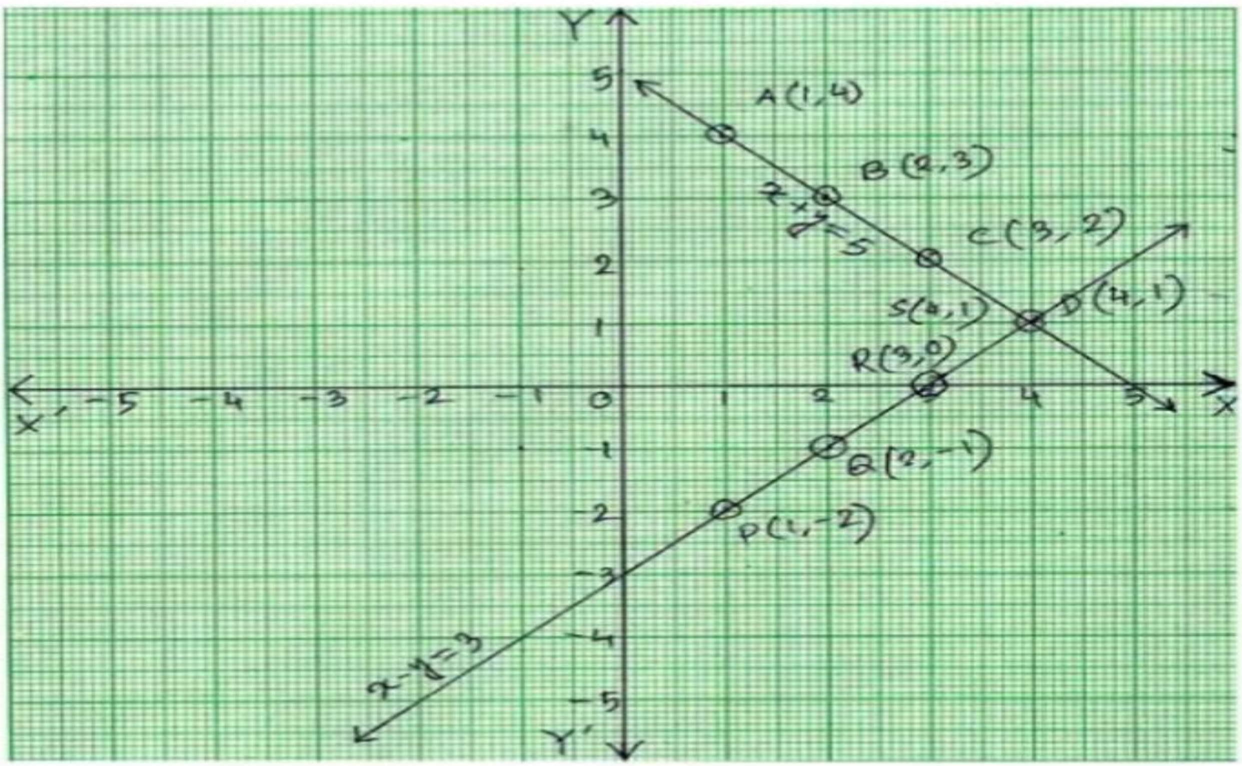
$x + y = 5$

0	1	2	3	4
y	4	3	2	1

$x - y = 3$

0	1	2	3	4
y	-2	-1	0	1

Now plot the points of the above tables on the graph paper.



Thus, we g0t two straight lines intersecting each other at $(4, 1)$, which is the common point of the intersection. Therefore, $x = 4$ & $y = 1$ is the only solution for the given system. And this type of system is called independent because the two lines intersects at a single point $(4,\ 1)$.

Examples (2): solve the following System of Linear Equations using the above methods stated:

$3x - y = 2$ ----------- (1) and $9x - 3y = 6$ ----------- (2)

Solution: Using substitution method ,$3x - y = 2$ --- (1) becomes $x = \frac{2+y}{3}$ substituting x into equation (2) gives $9\left(\frac{2+y}{3}\right) - 3y = 6$ ------ (2) $\Longrightarrow 3(2 + y) - 3y = 6 \implies 0 = 0 \ \therefore y = 0.$

And this implies that $x = 0$ & $y = 0$. Thus the system has an infinite number of solutions.

Aliter: Using elimination method, $3x - y = 2$ ------ (1) $\times\ 3$ or $9x - 3y = 6$ --------- (1)

$-\ \ 9x - 3y = 6$ ----- (2) $\qquad -\ \ 9x - 3y = 6$ --------- (2)

$0x + 0y = 0 \quad \therefore\ y = 0$

And this implies that $x = 0$ & $y = 0$. Thus the system has an infinite number of solutions.

Aliter: Using graphical method, we will apply the trial and error method again to find the pairs of values of $(\, x, y\,)$ which will satisfy the given equations.

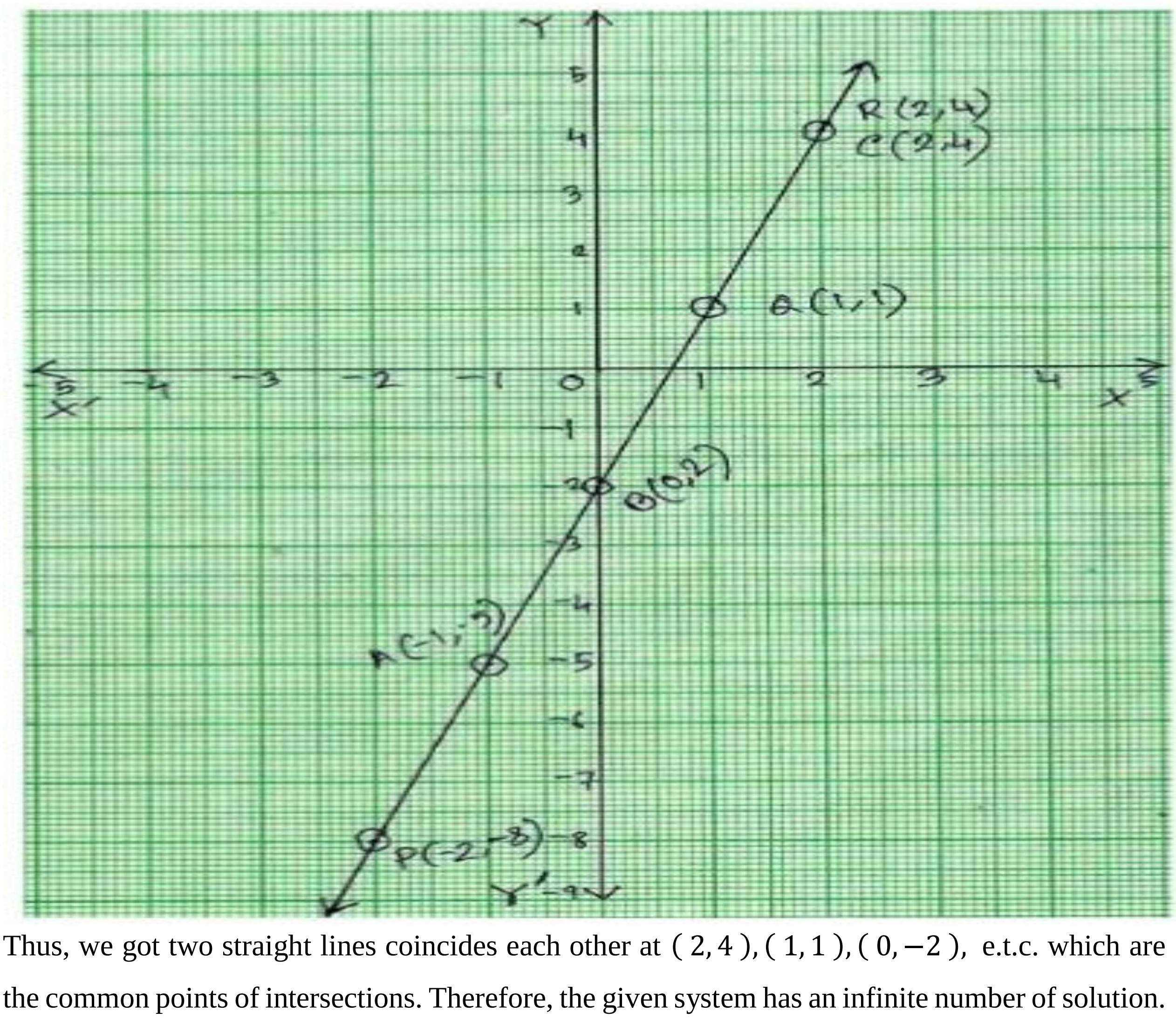
$3x - y = 2$

x	-1	1	3	4
y	-5	1	7	10

$9x - 3y = 6$

x	-2	0	2	4
y	-8	-2	4	10

Now plot the points of the above tables on the graph paper.



Thus, we got two straight lines coincides each other at $(\,2, 4\,), (\,1, 1\,), (\,0, -2\,)$, e.t.c. which are the common points of intersections. Therefore, the given system has an infinite number of solution. And this type of system is called dependent because the two lines coincides at different points.

Examples(3): solve the following System of Linear Equations using the above methods stated:
$2x - 3y = 5$ ----------- (1) and $6y - 4x = 3$ ----------- (2)

Solution: Using substitution method, $2x - 3y = 5$ -------- (1) becomes $x = \frac{5+3y}{2}$ substituting x into equation (2) gives $6y - 4\left(\frac{5+3y}{2}\right) = 3$ ------- (2) $\Rightarrow 6y - 2(5+3y) = 3 \Rightarrow 0 = 13$, which is false for every ordered pair (x, y). Thus this type of system has no solution.

Aliter: Using elimination method,

$$\begin{array}{rl} 4x - 6y = 10 & \text{-----------}\ (3) \\ +\ \ (-4x + 6y = 3\,) & \text{-----------}\ (4) \\ \hline 0x + 0y = 13 & \therefore\ 0 = 13 \end{array}$$

which is false for every ordered pair (x, y). Thus this type of system has no solution.

Aliter: Using graphical method, we will apply the trial and error method again to find the pairs of values of (x, y) which will satisfy the given equations.

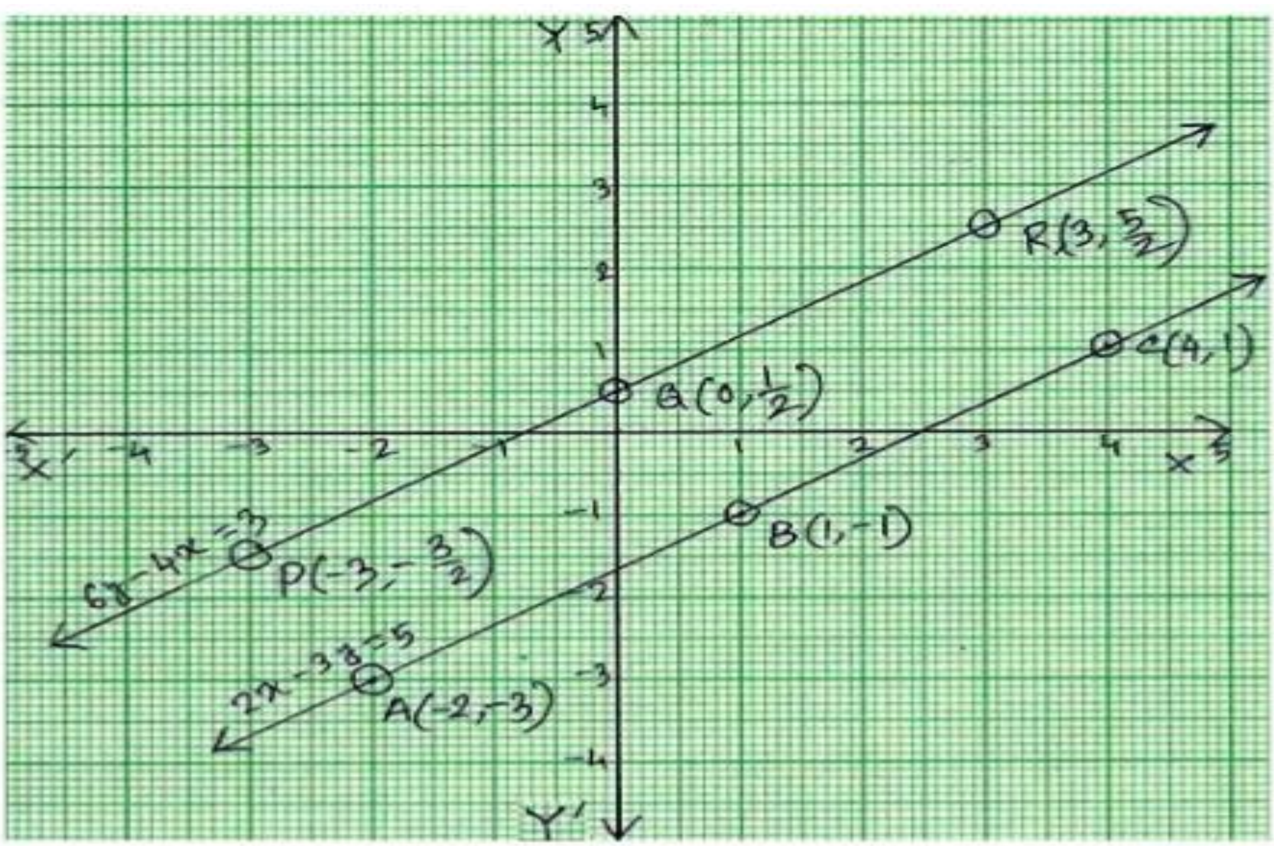
$2x - 3y = 5$

x	-2	1	4
y	-3	-1	1

$6y - 4x = 3$

x	-3	0	3
y	$-3/2$	$1/2$	$5/2$

Now plot the points of the above tables on the graph paper.



Thus, we got from the graph that the two straight lines are parallel to each other at $(-2,-3),(1,-1),(4,1)$, $(-3,-3/2),(0,1/2),(3,5/2)$. which are the common points of the intersections. Therefore, the given system has no common solutions. And this type of system is called inconsistent because the two lines have no common solutions.

Note: we conclude that to solve a pair of simultaneous equations graphically then the outcomes can be one of the following three:

Graphs	No. of solution	Classification
Unique solution if the lines intersect at a point (Non-Parallel lines)	One solution	Consistent
Infinitely many solutions if the two lines coincides (Identical lines)	many solution	Dependent
No solution if the two lines are parallel (Parallel lines).	No solution	Inconsistent

EXERCISE (16)

Solve the system of the linear equations below using the three ways & state its classification also

(1) $2x+3y=2$ and $x-2y=8$ (2) $4x+5y=13$ and $3x+y=-4$

(3) $2x+5y=16$ and $3x-7y=24$ (4) $7x-8y=9$ and $4x+3y=-10$

(5) $9x+2y=0$ and $4x-5y=17$ (6) $5x-6y=4$ and $3x+7y=8$

SYSTEM OF LINEAR EQUATIONS THREE VARIABLES (SIMULTANEOUS EQUATIONS)

In order to solve a system of equations in three variables, known as three-by-three systems, the primary goal is to eliminate one variable at a time to achieve back-substitution, A solution to a system of three equations in three variables (x,y,z) is called an ordered triple.

Example (1) Find the solution for the given system of equation in three variables below ?

$x-2y+3z=4---(1)$ $2x+y-4z=3---(2)$ & $-3x+4y-z=-2---(3)$

Solution: Multiplying (2) by 2, and adding it with (1) gives $5x-5z=10$ --------- (4)

Multiplying (1) by 2, and adding it with (3) gives $-x+5z=6$ --------- (5)

Solving (1) & (2) simultaneously we have $5x-5z=10$ --------- (4)

$+\ -x+5z=6$ --------- (5)

$4x+0y=16 \implies x=4$

Putting $x=4$ into eqn (4) gives $5(4)-5z=10 \implies z=2.$

Now substituting the values of x & z into eqn (1) gives $(4)-2y+3(2)=4 \implies y=3$

Thus, we have $x=4,\ y=3$ & $z=2$ as the only solution of the given system and is consistent.

Example (2) Determine the ordered triple for the given system of equation in three variable below

$x + 3y - 2z = 1 - - - (1) \quad x - 2y + 2z = -2 - - - (2) \;\&\; 2x + 3y - 4z = -4 - - - (3)$

Solution: Adding eqn. (1) & (2) gives $2x + y = -1$ --------- (4)

Multiplying (2) by 2, and adding it with (3) gives $4x - y = -8$ --------- (5)

Solving (1) & (2) simultaneously we have

$$\begin{array}{rl} & 2x + y = -1 \quad \text{--------- (4)} \\ + & \underline{4x - y = -8} \quad \text{--------- (5)} \\ & 6x + 0y = -9 \quad \Longrightarrow \quad x = -3/2 \end{array}$$

Putting $x = -3/2$ into eqn (4) gives $2(-3) + y = -1 \;\Longrightarrow\; y = 2.$

Now substituting the values of x & y into eqn (1) gives $(-3/2) + 3(2) - 2z = 1 \Longrightarrow z = 7/4$

Thus, we have $x = -3/2$, $y = 2$ & $z = 7/4$ as the required ordered triple of the given system

Note: Any system of three linear equations in three variables has either a unique solution, an infinite solution or no solution at all. The method of elimination can then proceed exactly as in the above example, and it is possible to simplify the process in such a way that, we don't have to write the variables. Using the above example we have:

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & -2 & 2 & -2 \\ 2 & 3 & -4 & -4 \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 1 & -2 & 2 & -2 \\ 2 & 3 & -4 & -4 \end{array}\right]$$

An array of numbers of this type is called a matrix. The coefficients of the variables is called the coefficient matrix, while the constant terms of the system is the augmented matrix.

EXERCISE (17)

Solve the system of the linear equations below using elimination method & state its classification

(1) $x - 2y - 3z = -1$, $2x + y + z = 6$, $x + 3y - 2z = 13$

(2) $x + 3y - z = -3$, $3x - y + 2z = 1$, $2x - y + z = -1$

(3) $4x - y + 3z = 6$, $-8x + 3y - 5z = -6$, $5x - 4y = -9$

MATRICES AND DETERMINANTS

In real life, we usually arrange our things in either horizontal or in vertical forms. For instance: we normally arrange desks or chairs in a classroom in the form mentioned above, or arrangement of soldiers for parade is always in vertical or horizontal form. These type of arrangements are also known as rows and columns respectively.

Definition: a matrix is a rectangular array of numbers that enclosed within curved bracket or square brackets.

In general, a matrix with m-rows & n-columns is said to have an order of $m \times n$ (read as m by n)

Geerally, we have
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}$$

Below are examples of matrices and their order (that is m by n illustration of the order).

$$\begin{array}{cccccc} 1-\begin{bmatrix}2 & 5 & 6\end{bmatrix} & 2-\begin{pmatrix}1\\0\\6\end{pmatrix} & 3-\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix} & 4-\begin{pmatrix}5 & 1 & 3\\2 & 4 & 3\end{pmatrix} & 5-\begin{bmatrix}1 & 0\\3 & -4\\2 & -1\end{bmatrix} & 6-\begin{bmatrix}1 & 2 & 3\\0 & -4 & 1\\6 & 5 & 4\end{bmatrix} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1\times 3 & 1\times 3 & 1\times 3 & 1\times 3 & 1\times 3 & 1\times 3 \end{array}$$

TYPES OF MATRIX

(1) A row matrix: This is a matrix with only one row of elements. Example: $A = \begin{bmatrix}2 & 5 & 6\end{bmatrix}$

(2) A column matrix: This is a matrix with only one column of elements. Example: $A = \begin{pmatrix}1\\0\\6\end{pmatrix}$

(3) Square matrix: This is a matrix with equal number of rows and columns of elements. Example: $A = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$ $B = \begin{bmatrix}1 & 2 & 3\\0 & -4 & 1\\6 & 5 & 4\end{bmatrix}$

(4) Zero or null matrix: This is a matrix whose elements are all zero. Example: $X = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

(5) Diagonal matrix: This is a square matrix whose elements are all zero except those along the leading diagonal. Example: $A = \begin{bmatrix}1 & 0 & 0\\0 & -5 & 0\\0 & 0 & 5\end{bmatrix}$

(6) A Unit (Identity) matrix: This is a square matrix whose elements in the leading diagonal are all one (that is unity). Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(7) Triangular matrix: This is a square matrix having zero elements below or above the leading diagonal. Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 8 & 1 \end{bmatrix}$

(8) Equal matrices: Two matrices are said to be equal if (1) they are of the same order (2) their corresponding elements are equal. Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \Longrightarrow \begin{matrix} a_{11} & = b_{12} \\ a_{21} & = b_{22} \end{matrix}$

For instance: given that $A = \begin{pmatrix} f & -2 \\ 4 & k \end{pmatrix}$ & $B = \begin{pmatrix} -1 & -2 \\ 4 & -3 \end{pmatrix}$ $\textit{the values of } f \,\&\, k \Longrightarrow \begin{matrix} f & = -1 \\ k & = -3 \end{matrix}$

Transpose of a matrix: This is a matrix obtained by interchanging the rows to columns. And it is denoted by A^T or A'. In general, if A is an (by n) matrix. Then A^T is an (by m) matrix. Example: if $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 8 & 1 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$

ADDITION AND SUBTRACTION OF MATRIX

Example(1): if $A = \begin{bmatrix} 2 & -4 & 3 \\ 5 & 2 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 4 & -2 \\ -3 & 3 & -1 \end{bmatrix}$ find (i) $A + B$ (ii) $B + A$ (iii) $A - A$ (iii) $B - A$?

(i) $A + B = \begin{bmatrix} 2 & -4 & 3 \\ 5 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & -2 \\ -3 & 3 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2+1 & -4+4 & 3-2 \\ 5-3 & 2+3 & 0-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 5 & -2 \end{bmatrix}$

(ii) $B + A = \begin{bmatrix} 1 & 4 & -2 \\ -3 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -4 & 3 \\ 5 & 2 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1+2 & 4-4 & -2+3 \\ -3+5 & 3+2 & -1-1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 5 & -2 \end{bmatrix}$

(ii) $A - B = \begin{bmatrix} 2 & -4 & 3 \\ 4 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 4 & -2 \\ -3 & 3 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2-1 & -4-4 & 3+2 \\ 5+3 & 1-3 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & -8 & 5 \\ 8 & -2 & 1 \end{bmatrix}$

(ii) $B - A = \begin{bmatrix} 1 & 4 & -2 \\ -3 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -4 & 3 \\ 5 & 2 & - \end{bmatrix} \Longrightarrow \begin{bmatrix} 1-2 & 4+4 & -2-3 \\ -3-5 & 3-2 & -1+1 \end{bmatrix} = \begin{bmatrix} -1 & 8 & -5 \\ -7 & 1 & 0 \end{bmatrix}$

MULTIPLICATION OF MATRIX

(a) Scalar Multiplication of matrix: The product of a matrix A and a scalar k (that is a number) is called the scalar product of the matrix A with its scalar k , and is denoted by kA.

Example(2): given that $A = \begin{bmatrix} 5 & 3 & 4 \\ 2 & -3 & 1 \end{bmatrix}$ find $3A$?

Sol: Here our k is 3 , $\Rightarrow\ 3A = 3\begin{bmatrix} 5 & 3 & 4 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 5 & 3\cdot 3 & 3\cdot 4 \\ 3\cdot 2 & 3\cdot -3 & 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 15 & 9 & 12 \\ 6 & -9 & 3 \end{bmatrix}$

Example(3): given that $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -3 \\ 5 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 6 & -1 \end{bmatrix}$, then find (i) $2A - 3B + C$? (ii) $4(B - 2C\ )$?

Sol: (i) $\Rightarrow 2\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & -3 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 0 & -9 \\ 15 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 19 \\ -5 & 2 \end{bmatrix}$

(ii) $4\left(\begin{bmatrix} 0 & -3 \\ 5 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 6 & -1 \end{bmatrix}\right) \Rightarrow 4\left(\begin{bmatrix} 0 & -3 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 12 & -2 \end{bmatrix}\right) = 4\begin{bmatrix} -2 & -7 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -8 & -18 \\ -28 & 12 \end{bmatrix}$

(b) Multiplication of matrix: Two matrices A & B can be multiplied together only if the number of columns of the matrix A is equal to the number of rows of the second matrix B. That is if $A = (\ 3\times 2\ )$ and $B == (\ 2\times 3\ ) \Rightarrow (\ 3\times 3\ )$

Example(4): given that $A = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 6 \end{bmatrix}$ then find (i) AB (ii) BA ?

(i)$AB = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 6 \end{bmatrix} = \begin{bmatrix} (2*1)+(3*-1) & (2*3)+(3*0) & (2*2)+(3*6) \\ (0*1)+(-2*-1) & (0*3)+(-2*0) & (0*2)+(-2*6) \\ (4*1)+(5*-1) & (4*3)+(5*0) & (4*2)+(5*6) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2-3 & 6+0 & 4+18 \\ 0+2 & 0+0 & 0-12 \\ 4-5 & 12+0 & 8+30 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 22 \\ 2 & 0 & -12 \\ -1 & 12 & 38 \end{bmatrix}$

(ii)$BA = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} (1*2)+(3*0)+(2*4) & (1*3)+(3*-2)+(2*5) \\ (-1*2)+(0*0)+(6*4) & (-1*3)+(0*-2)+(6*5) \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2+0+8 & 3-6+10 \\ -2+0+24 & -3+0+30 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 22 & 27 \end{bmatrix}$

Note that: matrix multiplication is not commutative. The above example proves the statement.

DETERMINANT OF $(\ 2\times 2\ )$ AND $(\ 3\times 3\ )$ MATRICES

Suppose A is a square matrix, then the determinant of the matrix A is always denoted by $|A|$. To find the determinant $(\ 2\times 2\ )$, multiply the element a_{11} & a_{22}, and then subtract $a_{12}\times a_{21}$ from it as in the below illustration.

- Let $A = \begin{bmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix, then the $|A| = \begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \end{vmatrix} \Rightarrow a_{11}a_{22} - a_{12}a_{21}$

Example(1): if $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} -3 & 1 \\ 2 & 4 \end{bmatrix}$ then find (a)$|A|$ (b)$|B|$ (c)$|A+B|$ (d)$|A| + |B|$

Solution: (a) $|A| = a_{11}a_{22} - a_{12}a_{21} \Longrightarrow (\,7 \times 2\,) - (\,3 \times 4\,) = 14 - 12 = 2.$

(b) $|B| \Longrightarrow (-3 \times 4\,) - (\,1 \times 2\,) = -12 - 2 = -14.$

(c)$|A+B| = \begin{vmatrix} 4 & 4 \\ 6 & 6 \end{vmatrix} \Longrightarrow (\,4 \times 6\,) - (\,4 \times 6\,) = 24 - 24 = 0.$ (d)$|A| + |B| = 2 - 14 = -12.$

■ Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a 3×3 matrix, then to evaluate its determinant, we first need to compute what we called the minors and cofactors of the matrix. Now if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the minors of the matrix is as follows:

$$A_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad A_{12} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad A_{21}, \quad A_{22}, \quad A_{23}, \quad A_{31}, \quad A_{32} \quad \& \quad A_{33}$$

While the cofactor of a matrix is a sign $(+ \ \ or \ -)$ attached to the minor A_{ij} to an element a_{ij} of the given matrix, that is $a_{ij} = (-1)^{i+j}\, A_{ij}$. & the cofactors are: $a_{11} = (-1)^{1+1}\, A_{11} \ \Longrightarrow +A_{11}$,

$a_{12} \Longrightarrow -A_{12}$, $a_{13} \Longrightarrow +A_{13,}$ $a_{21} \Longrightarrow -A_{21,}$ $a_{22} \Longrightarrow +A_{22}$, $a_{23} \Longrightarrow -A_{23,}$ $a_{31} \Longrightarrow +A_{31}$, $a_{32} \Longrightarrow -A_{32}$ & $a_{33} \Longrightarrow +A_{33}$.

Example(2): if $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \\ 0 & 7 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$ find the determinants of the matrices ?

Solution: $|A| \ \Longrightarrow 1 \begin{bmatrix} 0 & 5 \\ 7 & -1 \end{bmatrix} \ -0 \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} + 2 \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} = 1(0 - 35) - 0(-3 - 0) + 2(21 - 0)$

$|A| \ \Longrightarrow 1(-35) - 0(-3\,) + 2(21\,) = -35 - 0 + 42 \quad |A| \ \Longrightarrow \ 7.$

Solution: $|B| \Longrightarrow 2 \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix} \quad -3 \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \ - 4 \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = 2(-2 + 3) - 3(-1 - 9) - 4(-1 - 6)$ and $|B| \ \Longrightarrow 2(1) \ - 3(-10\,) \ - 4(-7\,) = \ 2 + 30 + 28 \quad |B| \ \Longrightarrow \ 60.$

Further: The alternative method of finding the determinant of a 3×3 matrix is by Sarrus rule

Example(3): if $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \\ 0 & 7 & -1 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$ find the determinants using Sarrus rule

(0) + (35) + (0) = 35

Solution: $|A| \implies \begin{matrix} 1 & 0 & 2 & 1 & 0 \\ 3 & 0 & 5 & 3 & 0 \\ 0 & 7 & -1 & 0 & 7 \end{matrix}$ $\quad : |A| = 42 - 35 = 7$

(0) + (0) + (42) = 42

(-24) + (-6) + (-3) = -33

Solution: $|B| \implies \begin{matrix} 2 & 3 & -4 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & -1 & -1 & 3 & -1 \end{matrix}$ $\quad : |A| = 27 - (-33) = 60$

(-4) + (27) + (4) = 27

INVERSE OF A MATRIX

Definition: Let A be a square matrix of order n, and suppose there exist another matrix B such that $AB = I \implies BA$. Then we called the matrx B as inverse of A and is denoted by A^{-1}. (read as A inverse). That is $AA^{-1} = I$, where I is an identity matrix of order n. To find the inverse of a given matrix, we first find the determinant and multiply with its adjoin, and it is achieve by swapping the position of a & b and putting negatives in front b & c, that is $A^{-1} = \frac{1}{|A|} \times Adj(A)$.

Example(1): Find the inverse of the matrix A if $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$?

Solution: Using $A^{-1} = \frac{1}{|A|} \times Adj(A)$, where $|A| = (7 \times 2) - (3 \times 4) = 14 - 12 = 2$, gives

$$A^{-1} = \frac{1}{|2|} \times \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ 2 & \frac{7}{2} \end{bmatrix}.$$

Example(2): Find the inverse of the matrix A if $A = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$?

Solution: Using $A^{-1} = \frac{1}{|A|} \times Adj(A)$, where $|A| = (3 \times 4) - (5 \times 1) = 12 - 5 = 7$, gives

$$A^{-1} = \frac{1}{|7|} \times \begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & -\frac{5}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{bmatrix}.$$

Note: To find the inverse of 3×3, we use $A^{-1} = \frac{1}{|A|} \times Adj(A)$ where the $Adj(A) = \left[cof\left(a_{ij}\right)\right]^T$

And if the square matrix has an inverse, then we called the matrix as an invertible matrix.

Example(3): Find the inverse of the matrix A if $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 5 \\ 0 & 7 & -1 \end{bmatrix}$?

(0) + (35) + (0) = 35

Solution: $|A| \implies \begin{matrix} 1 & 0 & 2 & 1 & 0 \\ 3 & 0 & 5 & 3 & 0 \\ 0 & 7 & -1 & 0 & 7 \end{matrix}$ $\qquad |A| = 42 - 35 = 7$

(0) + (0) + (42) = 42

Using $A^{-1} = \frac{adj(A)}{|A|} \times$, & $|A| = 7$, gives $A^{-1} = \frac{1}{|7|} \times \begin{bmatrix} -35 & 42 & 0 \\ 0 & 0 & 7 \\ 42 & -35 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 6 & 0 \\ 0 & 0 & 1 \\ 6 & -5 & 0 \end{bmatrix}$.

Example(4): Find the inverse of the matrix A if $A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{bmatrix}$?

(15) + (0) + (-12) = 3

Solution: $|A| \implies \begin{matrix} -1 & 3 & 1 & -1 & 3 \\ 2 & 5 & 0 & 2 & 5 \\ 3 & 1 & -2 & 3 & 1 \end{matrix}$ $\qquad |A| = 12 - 3 = 9$

(10) + (0) + (2) = 12

Using $A^{-1} = \frac{1}{|A|} \times Adj(A)$, where $|A| = 7$, gives $A^{-1} = \frac{1}{|9|} \times \begin{bmatrix} 10 & 14 & -15 \\ 12 & -5 & 2 \\ -13 & 0 & 22 \end{bmatrix} =$

$\begin{bmatrix} \frac{10}{9} & \frac{14}{9} & -\frac{15}{9} \\ \frac{12}{9} & -\frac{5}{9} & \frac{2}{9} \\ -\frac{13}{9} & 0 & \frac{22}{9} \end{bmatrix}$.

EXERCISE (18)

(1) Let $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & -2 \end{bmatrix}$ find if possible the following

(i)$A + B$ (ii)$B + A$ (iii)$A + C$ (iv)$A - B$ (v)$C - B$ (vi)AB (vii)BA (viii)BC (ix)CB (x)$A(B + C)$ (xi)$A(B - C)$ (xii)A^T (xiii)B^T (xiv)C^T (xv)$A^T + B^T$ (xvi)$A^T B^T$ (xvii)BB^T

(2) Find all the minors and cofactors where possible of the matrices below, the determinants of each matrix and determine whether they are invertible or not ?

Let $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$, $E = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$

$G = \begin{bmatrix} 5 & 1 & 3 \\ 0 & -1 & 2 \\ -2 & 4 & 6 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 8 \\ -4 & 1 & 3 \end{bmatrix}$, $I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $J = \begin{bmatrix} 1 & -5 & 0 \\ 4 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix}$

$K = \begin{bmatrix} 1 & 5 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathrm{L} = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $M = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 2 \\ -4 & 0 & 2 \end{bmatrix}$ $\mathrm{N} = \begin{bmatrix} 3 & 3 & 3 \\ -5 & 6 & 5 \\ 2 & 9 & 8 \end{bmatrix}$

SOLUTION OF SIMULTANEOUS EQUATIONS USING DETERMINANTS METHOD (CRAMMER'S RULE)

Suppose we have an equation $a_{11}x + a_{12}y = c_1 \quad ------ \quad (1)$ where $a_{11},\ a_{12}.\ a_{21}$

$a_{21}x + a_{22}y = c_1 \quad ------ \quad (2)$ a_{22} are all constants,

To solve for x & y using determinant method, we put the coefficients in the form of determinant first find the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, secondly $\Delta_x = \begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}$ & $\Delta_y = \begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}$

Then divide Δ_x by Δ to get x and divide Δ_y by Δ to get y, that is $\frac{\Delta_x}{\Delta} = \frac{c_1 a_{22} - c_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$

and $\frac{\Delta_y}{\Delta} = \frac{c_2 a_{11} - c_2 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$.

Example (1): Use determinant method to solve the following simultaneous equations

$2x + 3y = -2 \quad ------ \quad (1)$

$3x + 4y = -6 \quad ------ \quad (2)$

Solution: $\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1$ $\quad \Delta_x = \begin{vmatrix} -2 & 3 \\ -6 & 4 \end{vmatrix} = 10$ $\quad \Delta_y = \begin{vmatrix} 2 & -2 \\ 3 & -6 \end{vmatrix} = -6$

then $x = \frac{\Delta_x}{\Delta} = \frac{10}{-1} \Longrightarrow -10$ and $y = \frac{\Delta_y}{\Delta} = \frac{-6}{-1} \Longrightarrow 6$ Hence $x = -10$ and $y = 6$

Example (2): Use determinant method to solve the simultaneous equation using crammer's rule ?

$x + 5y = -3 \quad ------ \quad (1)$ and $2x - y = 5 \quad ------ \quad (2)$

Solution: $\Delta = \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} = -1 - 10 = -11$ $\quad \Delta_x = \begin{vmatrix} -3 & 5 \\ 5 & -1 \end{vmatrix} = -22$ $\quad \Delta_y = \begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix} = 11$

then $x = \frac{\Delta_x}{\Delta} = \frac{-22}{-11} \Longrightarrow 2$ and $y = \frac{\Delta_y}{\Delta} = \frac{11}{-11} \Longrightarrow -1$ Hence $x = 2$ and $y = -1$

Example (3): Solve the following simultaneous equation by the method of crammer's rule ?

$x + 3y - 2z = 1 --- (1)$ $\quad x - 2y + 2z = -2 --- (2)$ & $2x + 3y - 4z = -4 --- (3)$

(8) + (6) + (-12) = 2

Solution: $\Delta = \begin{vmatrix} 1 & 3 & -2 \\ 1 & -2 & 2 \\ 2 & 3 & -4 \end{vmatrix} \Rightarrow \begin{matrix} 1 & 3 & -2 & 1 & 3 \\ 1 & -2 & 2 & 1 & -2 \\ 2 & 3 & -4 & 2 & 3 \end{matrix}$ $\quad : \Delta = 14 - (2) = 12$

(8) + (12) + (-6) = 14

(-16) + (6) + (-36) = -46

$\Delta_x = \begin{vmatrix} 1 & 3 & -2 \\ 3 & -2 & 2 \\ -4 & 3 & -4 \end{vmatrix} \Rightarrow \begin{matrix} 1 & 3 & -2 & 1 & 3 \\ 3 & -2 & 2 & 3 & -2 \\ -4 & 3 & -4 & -4 & 3 \end{matrix}$ $\quad \Delta_x = -34 - (-46) = 12$

(8) + (-24) + (-18) = -34

(-12) + (-8) + (-4) = -24

$\Delta_y = \begin{vmatrix} 1 & 1 & -2 \\ 1 & 3 & 2 \\ 2 & -4 & -4 \end{vmatrix} \Rightarrow \begin{matrix} 1 & 1 & -2 & 1 & 1 \\ 1 & 3 & 2 & 1 & 3 \\ 2 & -4 & -4 & 2 & -4 \end{matrix}$ $\quad \Delta_y = 0 - (-24) = 24$

(-12) + (4) + (8) = 0

(-4) + (9) + (-12) = -7

$\Delta_z = \begin{vmatrix} 1 & 3 & 1 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} \Rightarrow \begin{matrix} 1 & 3 & 1 & 1 & 3 \\ 1 & -2 & 3 & 1 & -2 \\ 2 & 3 & -4 & 2 & 3 \end{matrix}$ $\quad \Delta_y = 29 - (-7) = 36$

(8) + (18) + (3) = 29

Then $x = \frac{\Delta_x}{\Delta} = \frac{12}{11} \Rightarrow 1,\ y = \frac{\Delta_y}{\Delta} = \frac{24}{12} \Rightarrow 2,\ z = \frac{\Delta_z}{\Delta} = \frac{36}{12} \Rightarrow 3 \quad \therefore\ x = 1\ \ y = 2\ \&\ z = 3$

Example (4): Solve the following simultaneous equation by the method of crammer's rule ?

$x + y + z - 2 = 0 \quad ------ \quad (1) \qquad \Rightarrow \quad x + y + z = 2 \quad ------ \quad (1)$

$2x - 3y - z + 1 = 0 \quad ------ \quad (2) \qquad \Rightarrow \quad 2x - 3y - z = -1 \quad ------ \quad (2)$

$3x + 2y - 4z - 27 = 0 ------ \quad (3) \qquad \Rightarrow \quad 3x + 2y - 4z = 27 \quad ------ \quad (3)$

$(-9) + (-2) + (-8) = -19$

Solution: $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 3 & 2 & -4 \end{vmatrix} \Rightarrow \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -3 & -1 & 2 & -3 \\ 3 & 2 & -4 & 3 & 2 \end{matrix}$ $\quad : \Delta = 13 - (-19) = 32$

$(12) + (-3) + (4) = 13$

$\Delta_x = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -3 & -1 \\ 27 & 2 & -4 \end{vmatrix} \Rightarrow 76 \quad \Delta_y = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 3 & 27 & -4 \end{vmatrix} \Rightarrow 98 \quad \Delta_z = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ 3 & 2 & 27 \end{vmatrix} \Rightarrow -110$

Then $= \frac{\Delta_x}{\Delta} = \frac{76}{32} \Rightarrow \frac{19}{8}$, $y = \frac{\Delta_y}{\Delta} = \frac{98}{32} \Rightarrow \frac{49}{16}$, $z = \frac{-110}{16} \Rightarrow \frac{-55}{16}$ $\therefore x = \frac{19}{8}$, $y = \frac{49}{16}$ & $z = \frac{-55}{16}$

EXERCISE (19)

Use the determinant method (crammer's rule) to solve the following system of equations below

(1) $2x + 3y = 2$ and $x - 2y = 8$ (2) $4x + 5y = 13$ and $3x + y = -4$

(3) $2x + 5y = 16$ and $3x - 7y = 24$ (4) $7x - 8y = 9$ and $4x + 3y = -10$

(5) $9x + 2y = 0$ and $4x - 5y = 17$ (6) $5x - 6y = 4$ and $3x + 7y = 8$

(7) $x - 2y - 3z = -1$
$2x + y + z = 6$
$x + 3y - 2z = 13$

(8) $x + 3y - z = -3$
$3x - y + 2z = 1$
$2x - y + z = -1$

(9) $4x - y + 3z = 6$
$-8x + 3y - 5z = -6$
$5x - 4y = -9$

SEQUENCE AND SERIES

A sequence (also called progression) is an ordered list of numbers such that each number can be obtained from the previous number according to some rule.

Forinstance: (i) 1, 4, 9, 16, $\cdot \cdot \cdot$ (ii) 7, 9, 11, $\cdot \cdot \cdot$ (iii) 1, 2, 4, 8, 16, $\cdot \cdot \cdot$

Each of these is a sequence, and the numbers are called as terms or an elements. However, if we wish to restrict our attention to a limited number of terms of a sequence, we could write: : (iv) 1, 4, 9, 16, $\cdot \cdot \cdot$, 144 (v) 7, 9, 11, $\cdot \cdot \cdot$, 31 (vi) 1, 2, 4, 8, 16, $\cdot \cdot \cdot$, 40 thus these numbers above can also be represented as a $a_1, a_2, \cdot \cdot \cdot, a_n$. Then such of these sequences in (i), (ii) & (iii) are said to be finite sequence, while (iv), (v) & (vi) are said to be infinite sequence.

LISTING THE TERMS OF A SEQUENCE.

Example (1): List all the terms of the given sequence $a_n = n^2$ for the range $1 \leq n \leq 5$?

Solution: $n^2 = 1^2,\ 2^2,\ 3^2,\ 4^2,\ 5^2 \implies 1,\ 4,\ 9,\ 16,\ 25.$

Example (2): List all the terms of the given sequence $a_n = \frac{1}{n+2}$ for the range $1 \leq n \leq 7$?

Solution: $\frac{1}{n+2} = \frac{1}{1+2}, \frac{1}{2+2}, \frac{1}{3+2}, \frac{1}{4+2}, \frac{1}{5+2}, \frac{1}{6+2}, \frac{1}{7+2} \implies \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}$

Example (3): List all the first five terms of the given infinite sequence $a_n = \frac{(-1)^n}{2^{n+1}}$?

Solution: $\frac{(-1)^n}{2^{n+1}} = \frac{(-1)^n}{2^{n+1}}, \frac{(-1)^n}{2^{n+1}}, \frac{(-1)^n}{2^{n+1}}, \frac{(-1)^n}{2^{n+1}}, \frac{(-1)^n}{2^{n+1}},$